HOLOMORPHIC FUNDAMENTAL SEMIGROUP OF RIEMANN DOMAINS

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Abstract. Let (W,Π) be a Riemann domain over a complex manifold M and w_0 be a point in W. Let $\mathbb{D}$ be the unit disk in $\mathbb{C}$ and $\mathbb{T}=\partial\overline{\mathbb{D}}$. Consider the space $\mathcal{S}_{1,w_0}(\overline{\mathbb{D}},W,M)$ of continuous mappings f of $\mathbb{T}$ into W such that $f(1)=w_0$ and $\Pi\circ f$ extends to a holomorphic on $\mathbb{D}$ mapping $\hat{f}$. Mappings $f_0,f_1\in\mathcal{S}_{1,w_0}(\overline{\mathbb{D}},W,M)$ are called *h-homotopic* if there is a continuous mapping f_t of $[0,1]$ into $\mathcal{S}_{1,w_0}(\overline{\mathbb{D}},W,M)$. Clearly, the h-homotopy is an equivalence relation and the equivalence class of $f\in\mathcal{S}_{1,w_0}(\overline{\mathbb{D}},W,M)$ will be denoted by $[f]$ and the set of all equivalence classes by $\eta_1(W,M,w_0)$.

There is a natural mapping $\iota_1:\eta_1(W,M,w_0)\to\pi_1(W,w_0)$ generated by assigning to $f\in\mathcal{S}_{1,w_0}(\overline{\mathbb{D}},W,M)$ its restriction to $\mathbb{T}$. We introduce on $\eta_1(W,M,w_0)$ a binary operation $\star$ which induces on $\eta_1(W,M,w_0)$ a structure of a semigroup with unity. Moreover, $\iota_1([f_1]\star[f_2])=\iota_1([f_1])\cdot\iota_1([f_2])$, where $\cdot$ is the standard operation on $\pi_1(W,w_0)$. Then we establish standard properties of $\eta_1(W,M,w_0)$ and provide some examples. In particular, we completely describe $\eta_1(W,M,w_0)$ when W is a finitely connected domain in $M=\mathbb{C}$ and Π is an identity. In particular, we show for a general domain $W\subset\mathbb{C}$ that $[f_1]=[f_2]$ if and only if $\iota_1([f_1])=\iota_1([f_2])$.

1. Introduction

Let $\mathcal{S}$ be a class of holomorphic mappings imbedded into a topological space $\mathcal{C}$ of continuous mappings and endowed with the relative topology. Mappings f_0 and f_1 are called h-homotopic if there is a continuous path in $\mathcal{S}$ connecting these mappings. The h-homotopy is an equivalence relation and the set of equivalence classes will be denoted by $\mathcal{H}$. We are interested in the structure of $\mathcal{H}$ and, in particular, in the relation between h-homotopy and topological homotopy.

Until recently not too much was known about the h-homotopy. It was not even clear how to approach the related problems. But in 1989 M. Gromov published the paper [G], where he introduced elliptic manifolds and proved that the homotopic Oka principle holds for holomorphic mappings into elliptic manifolds. This principle says that h-homotopy is equivalent to topological homotopy. F. Forsnerič and his colleagues expanded studies in this direction and their results can be found in [F]. However, elliptic or Oka manifolds are non-hyperbolic and, in general, the homotopic Oka principle fails.

Another rather old appearance of the h-homotopy is the Kontinuitätssatz or continuity principle of H. Kneser (1932). To state it we need some notation. Let $\mathbb{D}$ be the unit disk, $\mathbb{T}=\partial\mathbb{D}$ and let $\mathcal{A}(\overline{\mathbb{D}},\mathbb{C}^n)$ be the space of all continuous mappings of $\overline{\mathbb{D}}$ into $\mathbb{C}^n$ holomorphic on $\mathbb{D}$. For a domain $W\subset\mathbb{C}^n$ we set $\mathcal{S}=\mathcal{S}(\overline{\mathbb{D}},W,\mathbb{C}^n)$ to

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be the set of all $f \in \mathcal{A}(\overline{\mathbb{D}}, \mathbb{C}^n)$ such that $f(\mathbb{T}) \subset W$. The continuity principle states that if W is a domain of holomorphy, then $f \in \mathcal{S}$ is h-homotopic in $\mathcal{S}$ to a constant mapping if and only if $f(\overline{\mathbb{D}}) \subset W$.

The h-homotopy on $\mathcal{S}(\overline{\mathbb{D}}, W, \mathbb{C}^n)$ is an equivalence relation and in [Jo] the mappings from the equivalence class of the constant mapping were used to construct the envelope of holomorphy of W, which is generally non-schlicht or a Riemann domain. In some sense this paper was an expansion of the continuity principle. In [LP] the whole set of equivalence classes was used to find plurisubharmonic subextensions and the mappings in the equivalences classes were used to fill the holes in W.

In this paper we continue the study of the space $\mathcal{S}(\overline{\mathbb{D}}, W, \mathbb{C}^n)$ with a twist. More precisely, let (W, Π) be a Riemann domain over a complex manifold M and w_0 be a base point in W. As a class $\mathcal{S}$ we consider the space $\mathcal{S}_{1,w_0}(\overline{\mathbb{D}}, W, M)$ of continuous mappings f of $\mathbb{T}$ into W such that $f(1) = w_0$ and $\Pi \circ f$ extends to a holomorphic mapping $\hat{f}$ of $\mathbb{D}$ into M endowed with the natural topology (see Section 2). The equivalence class of $f \in \mathcal{S}_{1,w_0}(\overline{\mathbb{D}}, W, M)$ will be denoted by $[f]$ and the set of all equivalence classes by $\eta_1(W, M, w_0)$.

There is a natural mapping $\iota_1 : \eta_1(W, M, w_0) \to \pi_1(W, w_0)$ generated by assigning to $f \in \mathcal{S}_{1,w_0}(\overline{\mathbb{D}}, W, M)$ its restriction to $\mathbb{T}$. One of the achieved goals of this paper is to introduce on $\eta_1(W, M, w_0)$ a binary operation $\star$ and show that with this operation $\eta_1(W, M, w_0)$ becomes a semigroup with unity. Moreover, $\iota_1([f_1] \star [f_2]) = \iota_1([f_1]) \cdot \iota_1([f_2])$, where $\cdot$ is the standard operation on $\pi_1(W, w_0)$.

Since the standard concatenation of the restrictions of f_1 and f_2 to $\mathbb{T}$ cannot be realized as the boundary of an analytic disk we have to develop some machinery. This development starts in Section 2, where general facts are proved, and then continued in Section 3, where we study h-homotopy on planar compact sets. This allows us to introduce in Section 4 the $\star$ operation. Section 5 contains examples of $\eta_1(W, M, w_0)$ when W is an annulus in the complex plane or Riemann sphere and Section 6 is devoted to major properties of $\eta_1(W, M, w_0)$.

Finally, in Section 7 we completely describe $\eta_1(W, M, w_0)$ when W is a finitely connected domain in $M = \mathbb{C}$ and Π is an identity. In particular, we show for a general domain $W \subset \mathbb{C}$ that $[f_1] = [f_2]$ if and only if $\iota_1([f_1]) = \iota_1([f_2])$ and this manifests the homotopic Oka principle in a hyperbolic case.

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2. Basic facts

A *Riemann domain* over a complex manifold M is a pair (W, Π), where W is a path connected Hausdorff complex manifold and Π is locally biholomorphic mapping of W into M. Let $\hat{\rho}$ be a Riemann metric on M. The mapping Π lifts this metric to W as ρ.

Let N be another complex manifold and let K be a compact set in N. Suppose that a set $K' \subset K$ and $\phi : K' \to M$ is a continuous mapping. We denote by $\mathcal{A}_\phi(K, M)$ the set of all continuous mappings of K into M which are holomorphic on the interior K^o of K and are equal to ϕ on K'. If the set K' is empty then we denote $\mathcal{A}_\phi(K, M)$ by $\mathcal{A}(K, M)$.

If $B \subset K$ is a compact set containing ∂K, $K' \subset B$, then by $\mathcal{S}_\phi(B, K, W, M)$ we denote the set of all continuous mappings f of B into W such that $f = \phi$ on K'

and there is a mapping $\hat{f} \in \mathcal{A}(K, M)$ coinciding with $\Pi \circ f$ on ∂K. If the set K' is empty then we denote $\mathcal{S}_\phi(B, K, W, M)$ by $\mathcal{S}(B, K, W, M)$ and if $B = \partial K$ then we will write $\mathcal{S}_\phi(K, W, M)$ for $\mathcal{S}_\phi(\partial K, K, W, M)$. Note that if $W = M$ and Π is an identity, then $\mathcal{S}_\phi(B, K, W, M) = \mathcal{A}_\phi(K, M)$.

If the set K is compact and has a non-empty boundary, then the mapping $\hat{f}$ is unique due to the following proposition which allows us to define the mapping $\hat{\Pi} : \mathcal{S}_\phi(B, K, W, M) \to \mathcal{A}_{\hat{\phi}}(K, M)$, where $\hat{\phi} = \Pi \circ \phi$, as $\hat{\Pi}(f) = \hat{f}$.

Proposition 2.1. *Let $K \subset N$ be a compact set with a non-empty boundary. If $f, g \in \mathcal{A}(K, M)$ are equal on ∂K, then they are equal on K.*

Proof. If K_1 is a connected component of K^o and $z_0 \in \partial K_1$, then $z_0 \in \partial K$ and $f(z_0) = g(z_0)$. We can find neighborhoods U of z_0 and V of $f(z_0)$ biholomorphic to unit balls such that $f(U)$ and $g(U)$ lie in V. Let ϕ be a biholomorphic mapping of V onto the unit ball B.

Let $f_1 = \phi \circ f$ and $g_1 = \phi \circ g$ be mappings on some connected component E of $K_1 \cap U$. The function $u(z) = \log \|f_1 - g_1\|$, considered as a function with values at $\mathbb{R} \cup \{-\infty\}$, is subharmonic on E, continuous on $\overline{E}$, and is equal to $-\infty$ on $\partial E \cap \partial K$ which has non-empty relative interior in ∂E. Since the set of irregular boundary points on ∂E is polar (see [H]), the set of regular points is dense in the boundary and, consequently, for the harmonic measure μ_z relative to E and $z \in E$ we have $\mu_z(\partial E \cap \partial K) > 0$. Hence $u(z) = -\infty$ and we see that $f = g$ on E. Now standard arguments show that $f = g$ on K_1 and, after that, on K. □

We introduce the space $\mathcal{T}(N, W, M)$ of all triples (B, K, f), where $K \subset N$ is a compact set with a non-empty boundary, B is a compact set in K containing ∂K and $f \in \mathcal{S}(B, K, W, M)$. We define the topology on these space by choosing a system of neighborhoods. For this we introduce some Riemann metric d on N. If $(B, K, f) \in \mathcal{T}(N, W, M)$ and Φ is a continuous extension of $\hat{f}$ to N and $\varepsilon > 0$ we define a Φ, ε-neighborhood of (B, K, f) as a set of all triples $(A, L, g) \in \mathcal{T}(N, W, M)$ such that the Hausdorff distance between L and K and between A and B is less than ε and $\hat{\rho}(\hat{g}(z), \Phi(z)) < \varepsilon$ for all $z \in L$.

It is easy to verify that if U is a Φ, ε-neighborhood of (B, K, f), V is a Ψ, δ-neighborhood of (A, L, g) and $(C, D, h) \in U \cap V$, then there is a Λ, η-neighborhood of (C, D, h) lying in $U \cap V$. Hence our choice of neighborhoods defines a topology on $\mathcal{T}(N, W, M)$.

The set $\mathcal{S}_\phi(B, K, W, M) \subset \mathcal{T}(N, W, M)$ and we define the topology on this set as the topology relative to the topology imposed on $\mathcal{T}(N, W, M)$. We will frequently work with triples $(\partial K, K, f)$ and to simplify notation in this case we will write a pair (K, f) for $(\partial K, K, f)$. The space of all pairs $(K, f) \in \mathcal{T}(N, W, M)$ will be denoted by $\mathcal{S}^*(N, W, M)$.

The following example explains why we measure the distance between Φ and $\hat{g}$ and but not between f and g. Let $N = \mathbb{C}$, K is the close unit disk $\overline{\mathbb{D}}$ in $\mathbb{C}$, $W = \{z \in \mathbb{C} : 1 < |z| < 2\}$, $M = \mathbb{CP}^1$, and Π is the identity. The triples $(\partial K, K, f)$ and $(\partial K, K, g)$, where

$$f(\zeta) = \zeta \text{ and } g(\zeta) = \zeta + \frac{\varepsilon}{\zeta}$$

are close on ∂K when ε is small but $\hat{f}$ and $\hat{g}$ are not close in $\mathcal{A}(K, M)$.

We will use the notation $\mathcal{T}(N,M,M)$ when $W=M$ and Π is an identity. We define the mapping $\hat{\Pi}_1$ of $\mathcal{T}(N,W,M)$ into the set $\mathcal{T}(N,M,M)$ as $\hat{\Pi}_1(B,K,f)=(B,K,\hat{\Pi}(f))$. It follows immediately from the definitions of the topologies involved that the mapping $\hat{\Pi}_1$ is continuous.

Lemma 2.2. *Let $(B,K,f)\in\mathcal{T}(N,W,M)$ and let Φ be a continuous extension of $\hat{f}$ to N.*

1. *There is $\varepsilon_0>0$ such that for every $0<\varepsilon<\varepsilon_0$ the mapping $\hat{\Pi}_1$ maps the Φ,ε-neighborhood of (B,K,f) homeomorphically onto the Φ,ε-neighborhood of $(B,K,\hat{f})$.*
2. *There is $\delta_0>0$ such that if (A,L,g) lies in the Φ,δ_0-neighborhood of (B,K,f), then g can be extended to a holomorphic mapping $\widetilde{g}$ into W of the δ_0-neighborhood V of A in L and $\Pi\circ\widetilde{g}=\hat{g}$ on V.*
3. *There is $\delta_1>0$ such that if $(\partial L,L,g)$ lies in the Φ,δ_1-neighborhood of (B,K,f) and $K\subset L$, then g can be extended to a holomorphic mapping $\widetilde{g}$ into W of the δ_1-neighborhood V of $\partial L\cup B$ in L, the triple $(\partial L\cup B,L,\widetilde{g})$ lies in in the Φ,δ_1-neighborhood of (B,K,f) and $\Pi\circ\widetilde{g}=\hat{g}$ on V.*

Proof. (1) We define the mapping $\Pi_0: N\times W\to N\times M$ as $\Pi_0(z,w)=(z,\Pi(w))$. It is easy to see that there is a η-neighborhood U of the graph of f on B such that the restriction of Π_0 to U is a homeomorphism of U onto a neighborhood V of the graph of $\hat{f}$ on B. There is $\delta>0$ such that if $(x,y)\in N\times M$ and there is a point $z\in B$ such that $d(x,z)<\delta$ and $\hat{\rho}(y,\hat{f}(z))<\delta$, then $(x,y)\in V$.

Let us take $\varepsilon_0>0$ such that $\varepsilon_0<\delta/2$ and $\hat{\rho}(\hat{f}(z),\Phi(x))<\delta/2$ when $d(x,z)<\varepsilon_0$. If $\varepsilon<\varepsilon_0$ and (A,L,g) is in the Φ,ε-neighborhood of $(B,K,\hat{f})$ in $\mathcal{T}(N,M,M)$, then for any $x\in A$ there is $z\in B$ such that $d(x,z)<\varepsilon$. Hence $\hat{\rho}(g(x),\hat{f}(z))\le\hat{\rho}(g(x),\Phi(x))+\hat{\rho}(\Phi(x),\hat{f}(z))<\delta$. Thus the points $(z,g(z))$, $z\in A$, are in V. Then we can define the mapping $h: B\to W$ as $h(z)=P_W(\Pi_0^{-1}(z,g(z)))$, where P_W is the projection of $N\times W$ onto W. Clearly, $\hat{h}=g$, the triple $(A,L,h)\in\mathcal{T}(N,W,M)$ and $\hat{\Pi}_1(A,L,h)=(A,L,g)$. Moreover, (A,L,h) is in the Φ,ε-neighborhood of (B,K,f) in $\mathcal{T}(N,W,M)$.

If the triple (A,L,g) is in the Φ,ε-neighborhood of (B,K,f) in $\mathcal{T}(N,W,M)$, then $\hat{\Pi}_1(A,L,g)$ is in the Φ,ε-neighborhood of $(B,K,\hat{f})$ in $\mathcal{T}(N,M,M)$. Hence, $\hat{\Pi}_1$ is a bijection of the Φ,ε-neighborhood of (B,K,f) onto the Φ,ε-neighborhood of $(B,K,\hat{f})$. Since the continuity of $\hat{\Pi}_1^{-1}$ is easy to verify we proved (1).

(2) We take $\delta_0=\varepsilon_0/4$, where ε_0 was defined in (1). For (A,L,g) in the Φ,δ_0-neighborhood of (B,K,f) we take as C the closed δ_0-neighborhood of A in L. Then the triple $(C,L,\hat{g})$ is in the $\Phi,2\delta_0$-neighborhood of $(B,K,\hat{f})$ and by (1) there is $(C,L,\widetilde{g})$ in the $\Phi,2\delta_0$-neighborhood of (B,K,f) such that $\Pi\circ\widetilde{g}=\hat{g}$ on C and $\widetilde{g}=g$ on A.

(3) The proof follows the same line of argument as in (1) using the homeomorphism Π_0 and will be omitted. □

The following result that allows us to lift mappings from M to W is an immediate consequence of the lemma above.

Corollary 2.3. *For every $(B,K,f)\in\mathcal{T}(N,W,M)$ there is $\varepsilon>0$ such that for any continuous path $(A_t,L_t,\hat{g}_t)$ in the Φ,ε-neighborhood of $(B,K,\hat{f})$ there is a*

unique continuous path (A_t, L_t, g_t) in the Φ, ε-neighborhood of (B, K, f) such that $\hat{\Pi}_1(A_t, L_t, g_t) = (A_t, L_t, \hat{g}_t)$.

The following lemma establishes some sort of "convexity" of Φ, ε-neighborhoods.

Lemma 2.4. *Suppose that $f \in \mathcal{S}(B, K, W, M)$ and the graph $\Gamma_K^{\hat{f}}$ of $\hat{f}$ on K has a Stein neighborhood in $N \times M$. For every $\varepsilon > 0$ there is $\delta > 0$ such that if triples (A, L, g_0) and (A, L, g_1) lie in the Φ, δ-neighborhood of (B, K, f) in $\mathcal{T}(N, W, M)$, then there is a neighborhood X of the interval $[0, 1] \subset \mathbb{C}$ and a continuous mapping $G_\zeta : X \to \mathcal{S}(A, L, W, M)$ such that $G_0 = g_0$, $G_1 = g_1$, G_ζ lies in the Φ, ε-neighborhood of (K, f) and the mapping $\hat{G}_\zeta(z)$ is holomorphic in ζ for all $z \in L$. Moreover, if, additionally, a compact set $L' \subset A$ and $g_0|_{L'} = g_1|_{L'} = \phi$, then $G_\zeta|_{L'} = \phi$ for all $\zeta \in X$.*

Proof. Firstly, let us assume that M is Stein. We choose $\varepsilon > 0$ satisfying requirements of Lemma 2.2(1) and Corollary 2.3 and is so small that there is a compact set $Z \subset M$ such that $\hat{g}(L) \subset Z$ for any (A, L, g) in the Φ, ε-neighborhood V of $(B, K, \hat{f})$. Let F be an imbedding of M into $\mathbb{C}^N$ as a complex submanifold. By [GR, Theorem 8.C.8]) there are an open neighborhood U of $F(Z)$ in $\mathbb{C}^N$ and a holomorphic retraction P of U onto $U \cap F(M)$. Let $\widetilde{f} = F \circ \hat{f}$. Let us take $\sigma > 0$ so small that the σ-neighborhood of $\widetilde{f}(K)$ in $\mathbb{C}^N$ lies in U and for every z_1 in this neighborhood and any point $z_2 \in F(A)$ if $\|z_1 - z_2\| < \sigma$ then $\hat{\rho}(F^{-1}(P(z_1)), F^{-1}(P(z_2))) < \varepsilon$ and the interval $[z_1, z_2] \subset U$. There is $\delta > 0$ such that $\|F(w_1) - F(w_2)\| < \sigma$ when $w_1, w_2 \in A$ and $\hat{\rho}(w_1, w_2) < \delta$.

If (A, L, g_0) and (A, L, g_1) lie in the Φ, δ-neighborhood of (B, K, f) in $\mathcal{T}(N, W, M)$ and $\widetilde{g}_j = F \circ \hat{g}_j$, then $\hat{\rho}(\hat{g}_j(\zeta), \Phi(\zeta)) < \delta$ for all $\zeta \in L$. Hence, $\|\widetilde{f} - \widetilde{g}_j\| < \sigma$. If $\widetilde{h}_t = t\widetilde{g}_1 + (1 - t)\widetilde{g}_0$, $0 \le t \le 1$, then

$$\|\widetilde{h}_t(\zeta) - \widetilde{f}(\zeta)\| \le t\|\widetilde{f}(\zeta) - \widetilde{g}_1(\zeta)\| + (1 - t)\|\widetilde{f}(\zeta) - \widetilde{g}_0(\zeta)\| < \sigma$$

for all $\zeta \in L$ and $0 \le t \le 1$. Clearly, there is a neighborhood X of $[0, 1]$ in $\mathbb{C}$ such that this inequality holds for all $t \in X$. Thus $\widetilde{h}_t(L) \subset U$ and we can define the mappings $\hat{h}_t = F^{-1} \circ P \circ \widetilde{h}_t$. Clearly, $\hat{g}_0 = \hat{h}_0$ and $\hat{g}_1 = \hat{h}_1$ on K and by conditions on σ we have $\hat{\rho}(\hat{h}_t(\zeta), \Phi(\zeta)) < \varepsilon$ on L. Thus the holomorphic path $\hat{h}_t$ can be lifted to $\mathcal{S}(A, L, W, M)$ as G_t.

If, additionally, a compact set $L' \subset A$, $\phi : L' \to W$ is continuous and $g_0|_{L'} = g_1|_{L'} = \phi$, then the mappings $\widetilde{h}_t$ also are equal $\widetilde{g}_t$ on L'. Hence $G_t|_{L'} = \phi$ for all $t \in X$.

If M is not Stein but the graph $\Gamma_K^{\hat{f}}$ has a Stein neighborhood Y in $N \times M$, then we replace M with Y, W with $N \times W$, $f(\zeta)$ with $(\zeta, f(\zeta))$ and $g_j(\zeta)$ with $(\zeta, g_j(\zeta))$ for $j = 0, 1$. Then the same argument shows that the lemma holds. □

The following lemma allows us to shift slightly continuous paths in $\mathcal{S}^*(N, W, M)$.

Lemma 2.5. *Let (B_t, K_t, f_t), $0 \le t \le 1$, be a continuous path in $\mathcal{T}(N, W, M)$ such that the set $\hat{\Gamma} = \{(t, \zeta, \hat{f}_t(\zeta)), 0 \le t \le 1, \zeta \in K_t\}$ lies in some Stein domain $U \subset \mathbb{C} \times N \times M$. Let Φ_t be some continuous extension of the mapping $\hat{f}_t(\zeta)$ to $\mathbb{C} \times N$. For any $\varepsilon > 0$ there is $\delta > 0$ such that if (A_t, L_t, g_t) is a continuous path in $\mathcal{T}(N, W, M)$, w_t is a continuous path in W and $\xi_t \in A_t$ is a continuous path in N, $0 \le t \le 1$, and for all $0 \le t \le 1$ and $\zeta \in L_t$ the triples (A_t, L_t, g_t) lie in the Φ_t, δ-neighborhood of (B_t, K_t, f_t) and $\rho(g_t(\xi_t), w_t) < \delta$, then there is another continuous*

path (A_t, L_t, h_t) in $\mathcal{T}(N, W, M)$ such that $h_t(\xi_t) = w_t$ and $\hat\rho(\hat h_t(\zeta), \Phi(t, \zeta)) < \varepsilon$. Moreover, if $g_t(\xi_t) = w_t$ for some $0 \le t \le 1$ then $h_t \equiv g_t$.

Proof. Let F be an imbedding of U into $\mathbb{C}^p$ as a complex submanifold. By [GR, Theorem 8.C.8] there are an open neighborhood $Y \subset\subset \mathbb{C}^p$ of $F(\hat\Gamma)$ in $\mathbb{C}^p$ and a holomorphic retraction P of Y onto $F(U) \cap Y$. Let P_M be the projection of $\mathbb{C} \times N \times M$ onto M. We may assume that there is a constant $C \ge 1$ such that $\|F(t, \zeta, z) - F(t, \zeta, w)\| \le C\hat\rho(z, w)$ on U and $\hat\rho(P_M(F^{-1}(P(z_1)), P_M(F^{-1}(P(z_2)) \le C\|z_1 - z_2\|$ when $z_1, z_2 \in Y$. We take an open set $Y' \subset\subset Y$ containing $F(\hat\Gamma)$. There is $\sigma > 0$ such that $Y' + v \subset Y$ for any $v \in \mathbb{C}^p$ with $\|v\| < \sigma$. Let $U' = F^{-1}(Y' \cap F(U))$.

Let $\Gamma = \{(t, \zeta, f_t(\zeta)) \in \mathbb{C} \times N \times W : 0 \le t \le 1, \zeta \in B_t\}$. The set Γ is compact and there is a neighborhood Z of Γ such that the mapping $\Pi_2(t, \zeta, w) = (t, \zeta, \Pi(w))$ is a homeomorphism of Z onto an open set $\hat Z \subset \mathbb{C} \times N \times M$. For every $0 \le t \le 1$ and $\zeta \in B_t$ the point $(t, \zeta, \hat f_t(\zeta)) \in \hat Z$. Therefore there is $\eta > 0$ such that $(t, \zeta, z) \in \hat Z$ if $\rho_N(\zeta, \zeta_1) < \eta$ for some $\zeta_1 \in B_t$ and $\hat\rho(z, \Phi_t(\zeta)) < \eta$.

Given a continuous path (A_t, L_t, g_t) in $\mathcal{T}(N, W, M)$, a continuous path w_t in W and a continuous path ξ_t in N satisfying conditions of the lemma we presume, firstly, that δ is so small that for $0 \le t \le 1$ the paths $(t, \zeta, \hat g_t(\zeta))$, $\zeta \in L_t$, and $(t, \xi_t, \hat w_t)$, $\hat w_t = \Pi(w_t)$, lie in Y'. Hence we can define the mappings $\widetilde g_t(\zeta) = F(t, \zeta, \hat g_t(\zeta))$ of L_t into $\mathbb{C}^p$ and the path $\widetilde w_t = F(t, \xi_t, \hat w_t)$ in $\mathbb{C}^p$. Clearly, $\|\widetilde w_t - \widetilde g_t(\xi_t)\| < C\delta$. So if we require that $C\delta < \sigma$ then the path $\widetilde h_t(\zeta) = \widetilde g_t(\zeta) - \widetilde g_t(\xi_t) + \widetilde w_t$, $\zeta \in L_t$, lies in Y and we can define $\hat h_t(\zeta) = P_M(F^{-1}(P(\widetilde g_t(\zeta))))$. Note that $\hat h_t = \hat g_t$ if $g_t(\xi_t) = w_t$.

Since $\|\widetilde h_t(\zeta) - \widetilde g_t(\zeta)\| \le C\delta$ for $0 \le t \le 1$ and $\zeta \in L_t$ we see that $\hat\rho(\hat h_t(\zeta), \hat g_t(\zeta)) \le C^2\delta$. Hence $\hat\rho(\hat h_t(\zeta), \hat f_t(\zeta)) \le (1 + C^2)\delta$. So if we require that $(1 + C^2)\delta < \min\{\eta, \varepsilon\}$ then the points $(t, \zeta, \hat h_t(\zeta)) \in \hat Z$ when $0 \le t \le 1$ and $\zeta \in \partial L_t$. Let P_W be the projection of $\mathbb{C} \times N \times W$ onto W and $h_t(\zeta) = P_W \circ \Pi_2^{-1}(t, \zeta, \hat h_t(\zeta))$ for $\zeta \in \partial L_t$. Then $\Pi \circ h_t = \hat h_t$ and $h_t(\xi_t) = w_t$. □

We say that $f, g \in \mathcal{S}_\phi(B, K, W, M)$ are *h, ϕ-homotopic* or $f \sim_\phi^h g$ if there is a continuous path connecting f and g in $\mathcal{S}_\phi(B, K, W, M)$. The relation $\sim_\phi^h$ is evidently an equivalence and we will call the equivalence class of f by the *h-homotopic type relative to the base ϕ* of f and denote by $[f]_\phi$. The set of equivalence classes will be denoted by $\mathcal{H}_\phi[B, K, W, M]$ or $\mathcal{H}_\phi[K]$ and if $[f]_\phi = [g]_\phi$ then we say that f and g are *h, ϕ-homotopic* or *h-homotopic.*

As the following corollary shows the homotopic type is a continuous functions on $\mathcal{S}_\phi(B, K, W, M)$ provided the existence of Stein neighborhoods for the graphs.

Corollary 2.6. *Let $f, g \in \mathcal{S}_\phi(B, K, W, M)$ and the graphs of f and g have Stein neighborhoods in $N \times M$. If there are sequences $\{f_j\}$ and $\{g_j\}$ converging to f and g respectively in $\mathcal{S}_\phi(B, K, W, M)$ and such that $f_j \sim_\phi^h g_j$, then $f \sim_\phi^h g$.*

Proof. By Lemma 2.4 there is j_0 such that $f_j \sim_\phi^h f$ and $g_j \sim_\phi^h g$ when $j \ge j_0$. Since the relation $\sim_\phi^h$ is transitive we see that $f \sim_\phi^h g$. □

3. Homotopic types of holomorphic mappings of planar compact sets

Throughout this section K will denote a connected compact set in $\mathbb{C}$ with the connected complement. Let $\zeta_0 \in \partial K$, $K' = \{\zeta_0\}$, a base point $w_0 \in W$ and $\phi(\zeta_0) = w_0$. We will denote $\mathcal{S}_\phi(B, K, W, M)$ by $\mathcal{S}_{\zeta_0, w_0}(B, K, W, M)$. It is rather

difficult to describe the set $\mathcal{H}_\phi[B,K,W,M] = \mathcal{H}_{\zeta_0,w_0}[B,K,W,M]$ even in this case. To get some information we construct in this section a mapping of this set into the set $\mathcal{H}_{1,w_0}[\mathbb{T},\overline{\mathbb{D}},W,M]$. Two facts will help us to do this: firstly, by Corollary 4.4 in [P] any mapping $f \in \mathcal{A}(K,M)$ can be approximated by holomorphic mappings on neighborhoods of K and, secondly, by Theorem 3.1 in [P] the graph Γ_K^f of f on K has a basis of Stein neighborhoods in $\mathbb{C} \times M$.

Let D be a Jordan domain, i.e., a domain bounded by a Jordan curve (a homeomorphic image of a circle). Let $\zeta_0 \in \partial D$ and let ζ_1 be a point in D. We will associate with D, ζ_0 and ζ_1 a unique conformal mapping e_{D,ζ_0,ζ_1} of the unit disk $\mathbb{D}$ onto D which maps 1 to ζ_0 and 0 to ζ_1. If $g \in \mathcal{S}_{\zeta_0,w_0}(\overline{D},W,M)$ then we let $h_{D,\zeta_0,\zeta_1} = g \circ e_{D,\zeta_0,\zeta_1}$ and denote by $\{g,\zeta_0\}$ the equivalence class of h_{D,ζ_0,ζ_1} in $\mathcal{H}_{1,w_0}[\mathbb{T},\overline{\mathbb{D}},W,M]$. The choice of the point ζ_1 does not influence $\{g,\zeta_0\}$ because the group of conformal automorphisms of $\mathbb{D}$ with a fixed point on the boundary is contractible and in the future we will remove ζ_1 from notation.

We will need to construct continuous paths $(\partial D_t, \overline{D}_t, f_t)$. In general, it is much more difficult to shift compact sets than the mappings. But when D_t is a Jordan domain, then the notion of Radó continuity described below is very helpful.

Suppose that we have a family of Jordan domains $D_t \subset \mathbb{C}$, $0 \le t \le 1$, such that a neighborhood of a point ζ belongs to the intersection of all D_t. Such a family is *Radó continuous* if the family of conformal mappings ϕ_t of $\mathbb{D}$ onto D_t such that $\phi_t(0) = \zeta$ and $\phi_t'(0) > 0$ is continuous on $\overline{\mathbb{D}} \times [0,1]$. (By a theorem of Carathéodory the mappings ϕ_t extend to $\overline{\mathbb{D}}$ as its homeomorphisms onto $\overline{D}_t$.) A result of Radó (see [Ra] or [Go, Theorem II.5.2]) claims, in particular, that a family of Jordan domains $D_t \subset \mathbb{C}$ is Radó continuous if and only if for every $t_0 \in [0,1]$ there are homeomorphisms $\Psi(t,\zeta)$ of ∂D_{t_0} onto ∂D_t converging uniformly to identity on ∂D_{t_0} as $t \to t_0$.

Suppose that D_t, $0 \le t \le 1$, is a Radó continuous family of Jordan domains and ζ_t is a continuous path in $\mathbb{C}$ such that $\zeta_t \in \partial D_t$. Let ψ_t be conformal mappings of $\mathbb{D}$ onto D_t such that $\psi_t(0) = \zeta$ and $\psi_t(1) = \zeta_t$. Then this family is also continuous on $\overline{\mathbb{D}} \times [0,1]$. Indeed, if $0 \le t_0 \le 1$ then, rotating $\mathbb{C}$ if necessary, we may assume that $\psi_{t_0}'(0) > 0$ and $\psi_{t_0} = \phi_{t_0}$. If $\xi_t \in \partial\mathbb{D}$ and $\phi_t(\xi_t) = \zeta_t$, then $\xi_t \to 1$ as $t \to t_0$. Hence ψ_t differs from ϕ_t by a rotation by a small angle and this angle goes to 0 as $t \to t_0$.

As the proof of the following lemma demonstrates the notion of Radó continuity allows us to shift at least Jordan domains.

Lemma 3.1. *Let $(B,K,f) \in \mathcal{T}(\mathbb{C},W,M)$ and $w_0 \in W$. There is $\delta > 0$ such that if:*

1. *$D_0 \subset\subset D_1$ are Jordan domains and $K \subset D_1$;*
2. *$(\partial D_1 \cup B, \overline{D}_1, g_1)$ and $(\partial D_0, \overline{D}_0, g_0)$ lie in the Φ,δ-neighborhood of (B,K,f) in $\mathcal{T}(\mathbb{C},W,M)$;*
3. *$\zeta_0 \in \partial D_0$ and $\zeta_1 \in \partial D_1$ and $g_0(\zeta_0) = g_1(\zeta_1) = w_0$;*
4. *there is a continuous curve $\gamma : [0,1] \to \overline{D}_1 \setminus D_0$ of diameter less than δ and such that $\gamma(0) = \zeta_0$, $\gamma(1) = \zeta_1$ and $\gamma(t) \in D_1 \setminus \overline{D}_0$, $0 < t < 1$,*

then $\{g_1,\zeta_1\} = \{g_0,\zeta_0\}$.

Proof. For some $\varepsilon > 0$ and the triple (B,K,f) we choose $\eta > 0$ as δ in Lemma 2.4. Then for the chosen η let us choose $0 < \sigma < \eta$ so that we can use Lemma 2.5 with ε replaced by η and δ by σ.

Suppose that $\delta > 0$ is already chosen. If $\zeta \in \overline{D}_1 \setminus D_0$ then $d(\zeta, K) < \delta$ because $\overline{D}_1$ lies in the δ-neighborhood of K. But K lies in the δ-neighborhood of D_0 and, therefore, $d(\zeta, D_0) < 2\delta$. Since $\zeta \notin D_0$, $d(\zeta, \partial D_0) < 2\delta$ and, since ∂D_0 lies in the δ-neighborhood of B we see that $d(\zeta, B) < 3\delta$.

Our first requirement on δ is that 3δ should be less than δ_0 in Lemma 2.2(2). Then g_1 extends to $\overline{D}_1 \setminus D_0$. We will denote this extension also by g_1.

Let Θ be a conformal mapping of $D_1 \setminus \overline{D}_0$ onto the annulus $A(r_0, 1) = \{\zeta \in \mathbb{C} : r_0 < |\zeta| < 1\}$ mapping ∂D_0 onto the unit circle. We define the intermediate domains D_t as bounded domains with boundaries equal to $\Theta^{-1}(\{|\zeta| = (1 - r_0)t + r_0\})$. The domains D_t are simply connected and the family D_t is Radó continuous. To prove the latter statement we note that as homeomorphisms Ψ_t of ∂D_t onto ∂D_{t_0} we can take preimages under the mapping Θ of the radial correspondences between circles in $A(r_0, 1)$. We will reparameterize this family letting $D_t := D_s$, $\gamma(t) \in \partial D_s$, $t \in [0, 1]$. Then the new family is still Radó continuous.

Let us set $A_t = \partial D_t \cup (B \cap \overline{D}_t)$ and consider the path $(A_t, \overline{D}_t, h_t)$, where h_t are restriction of g_1 to A_t. The set $\overline{D}_t \subset \overline{D}_1$ and, therefore, lies in the δ-neighborhood of K. In its turn K lies in the δ-neighborhood of D_0 which lies in D_t. So the Hausdorff distance between $\overline{D}_t$ and K is less than δ. We know that A_t lies in in the 3δ-neighborhood of B. If $\zeta \in B \setminus D_t$ then $\zeta \in \overline{D}_1 \setminus D_0$ and, by above, $d(\zeta, \partial D_0) < 2\delta$. Hence $d(\zeta, A_t) < 2\delta$. So the Hausdorff distance between A_t and B is less than 3δ. Consequently, the path $(A_t, \overline{D}_t, h_t)$ lies in the $\Phi, 3\delta$-neighborhood of (B, K, f).

Our second requirement for δ is that $2\delta < \sigma$ and $\rho(g_1(\zeta), w_0) < \sigma$ when $\zeta \in \overline{D}_1 \setminus D_0$ and $d(\zeta, \zeta_1) < \delta$. Then $\rho(h_t(\gamma(t)), w_0) < \sigma$ and by Lemma 2.5 we can replace the path $(A_t, \overline{D}_t, h_t)$ with the path $(A_t, \overline{D}_t, p_t)$ in the Φ, η-neighborhood of (B, K, f) such that $p_t(\gamma(t)) = w_0$.

The triple $(A_0, \overline{D}_0, p_0) = (\partial D_0 \cup (B \cap D_0), \overline{D}_0, p_0)$ is the Φ, η-neighborhood of (B, K, f). So the triple $(\partial D_0, \overline{D}_0, p_0)$ is in the same neighborhood. By Lemma 2.4 there is a continuous path $(\partial D_0, \overline{D}_0, q_t)$ in the Φ, ε-neighborhood of (B, K, f) connecting $(\partial D_0, \overline{D}_0, p_0)$ and $(\partial D_0, \overline{D}_0, g_0)$ and such that $q_t(\zeta_0) = w_0$. Concatenation of these two paths provides a continuous path $(\partial G_t, \overline{G}_t, g_t)$ connecting $(\partial D_0, \overline{D}_0, g_0)$ and $(\partial D_1, \overline{D}_1, g_1)$ such that the family of Jordan domains G_t is Radó continuous. We let $\zeta_t = \gamma(t)$ on the first part of his path and $\zeta_t = \zeta_0$ on the second part.

By the theorem of Radó the path $(\mathbb{T}, \overline{\mathbb{D}}, g_t \circ e_{G_t, \zeta_t, \zeta_1})$ is also continuous. Hence, $\{g_1, \zeta_1\} = \{g_0, \zeta_0\}$. $\square$

Let $\gamma : [0, 1] \to \mathbb{C}$ be a continuous curve such that $\gamma(t) \in \mathbb{C} \setminus K$ when $t > 0$ and $\gamma(0) = \zeta_0$. Such curves will be called *access* curves to K at ζ_0. In the terminology of the prime ends theory it means that the point ζ_0 is accessible in $\mathbb{C} \setminus K$. If D is a domain which meets γ we let $\zeta_{D,\gamma} = \gamma(s_{D,\gamma})$, where $s_{D,\gamma} = \inf\{t : \gamma(t) \in \partial D\}$.

If D is a smooth Jordan domain containing K, D meets γ at ξ_1, a pair $(\overline{D}, g) \in \mathcal{S}^*(\mathbb{C}, W, M)$ and $g(\xi_1) = w_0$, then we say that the triple $(\overline{D}, g, \xi_1)$ is a Φ, ε-approximation of $f \in \mathcal{S}_{\zeta_0, w_0}(B, K, W, M)$ if $(\overline{D}, g)$ lies in the Φ, ε-neighborhood of (K, f). We will write $(\overline{D}, g)$ for $(\overline{D}, g, \zeta_{D,\gamma})$ and say that $(\overline{D}, g)$ is a Φ, ε-approximation of f with respect to γ.

The following proposition asserts the existence of Φ, ε-approximations for every Φ and ε.

Proposition 3.2. *Let $f \in \mathcal{S}_{\zeta_0,w_0}(B,K,W,M)$, let Φ be a continuous extension of $\hat{f}$ to $\mathbb{C}$ as a mapping to M and let γ be an access curve to K at ζ_0. Then for every $\varepsilon > 0$ there is a Φ,ε-approximation $(\overline{D}, g, \zeta_1)$ of f, where ζ_1 is any point in $\partial D \cap \gamma$.*

Proof. By Lemma 2.5 for every $\varepsilon > 0$ there is $\delta > 0$ such that if $(\overline{D}, h)$ lies in the Φ,δ-neighborhood of (K,f) and $\rho(h(\zeta_1), w_0) < \delta$ for some $\zeta_1 \in \partial D$, then there is a mapping $g \in \mathcal{S}_{\xi_1,w_0}(\overline{D}, W, M)$ such that $(\overline{D}, g)$ lies in the Φ,ε-neighborhood of (K,f). The set $\hat{\Gamma} = \{(t,\zeta,\hat{f}(\zeta)), 0 \le t \le 1, \zeta \in K\}$ has a Stein neighborhood in $\mathbb{C} \times N \times M$. By Corollary 4.4 from [P] for every $\delta > 0$ there is a smooth Jordan neighborhood D of K and a mapping $\hat{h} \in \mathcal{S}(\overline{D}, M, M)$ such that $(\overline{D}, \hat{h})$ lies in the Φ,δ-neighborhood of $(K, \hat{f})$. Taking $\delta < \varepsilon_0$ in Lemma 2.2(1) we can lift $\hat{h}$ to $\mathcal{S}(\overline{D}, W, M)$ as h.

We may assume that δ is so small that if ∂D meets γ at ζ_1 then $\rho(h(\zeta_1), w_0) < \delta$. Shifting h with Lemma 2.5 we get the triple $(\overline{D}, g, \zeta_1)$ providing the needed approximation. □

The following proposition asserts that if the pair $(\overline{D}, g)$ is a sufficiently good approximation of some $(B,K,f) \in \mathcal{S}_{\zeta_0,w_0}(B,K,W,M)$ then $\{g, \zeta_{D,\gamma}\}$ does not depend on D and g.

Proposition 3.3. *Let $f \in \mathcal{S}_{\zeta_0,w_0}(B,K,W,M)$ and let γ be an access curve to K at ζ_0. There is $\delta > 0$ such that if and $(\overline{D}_0, g_0)$ and $(\overline{D}_1, g_1)$ are Φ,δ-approximations of (K,f) such that ∂D_0 and ∂D_1 meet γ, then $\{g_0, \zeta_{D_0,\gamma}\} = \{g_1, \zeta_{D_1,\gamma}\}$.*

Proof. Let us take δ less than $\delta/2$ from Lemma 3.1 and ε from Proposition 3.2. Suppose that $s_{D_0,\gamma} \le s_{D_1,\gamma}$. We take a Jordan domain $D \subset\subset D_0 \cap D_1$ containing K such that the restriction of the curve γ to $[s_{D_0,\gamma}, s_{D_1,\gamma}]$ lies outside of $\overline{D}$. Let $t_1 = \sup\{t : \gamma(t) \in D\}$ and $\xi_1 = \gamma(t_1)$. Then the restriction γ_1 of the curve γ to $[t_1, s_{D_1,\gamma}]$ lies in $\overline{D}_1 \setminus D$. By Proposition 3.2 we can find a Φ,δ-approximation (D, g, ξ_1) of (K,f). By Lemma 3.1 $\{g, \xi_1\} = \{g_1, \zeta_{D_1,\gamma}\}$. If we replace γ_1 with the restriction γ_2 of the curve γ to $[t_1, s_{D_0,\gamma}]$ the same reasoning shows that $\{g, \xi_1\} = \{g_0, \zeta_{D_0,\gamma}\}$. □

Consequently, for $f \in \mathcal{S}_{\zeta_0,w_0}(K,W,M)$ there is a Φ,ε-neighborhood of (K,f) such that the class $\{g, \zeta_{D,\gamma}\}$ is the same for all pairs $(\overline{D}, g)$ in this neighborhood and it will be denoted by $[f,\gamma]$.

The following result shows that $[f,\gamma]$ continuously depends on (B,K,f).

Theorem 3.4. *Let $f \in \mathcal{S}_{\zeta_0,w_0}(B,K,W,M)$ and let γ be an access curve to K at ζ_0. For any continuous extension Φ of f there is $\eta > 0$ such that if a triple $(\partial L, L, g)$ lies in the Φ,η-neighborhood of (B,K,f) in $\mathcal{T}(\mathbb{C}, W, M)$ the point $\zeta_0 \in \partial L$, $g(\zeta_0) = w_0$ and γ is an access curve to L, then $[f,\gamma] = [g,\gamma]$.*

Proof. Let us take δ satisfying Lemma 3.1. We take $\eta < \min\{\delta_0, \delta_1, \delta/2\}$, where δ_0 and δ_1 are taken from Lemma 2.2, and find a pair $(\overline{D}_0, f_0)$, where D_0 is a Jordan domain containing K, in the Φ,η-neighborhood Y of (K,f) such that $\{f_0, \zeta_{D_0,\gamma}\} = [f,\gamma]$. By Lemma 2.2(2),(3) the mapping f_0 extends to the δ_0-neighborhood of B so that the triple $(\partial D_0 \cup B, \overline{D}_0, f_0)$ lies in the Φ,η-neighborhood of (B,K,f).

We assume that η is so small that the diameter of γ in D_0 is less than δ. Then we take a continuous extension Ψ of g and $\sigma > 0$ such that Ψ,σ-neighborhood V of (L,g) lies in Y. There is a pair $(\overline{D}_1, g_0) \in V$, where $D_1 \subset\subset D_0$ is a Jordan domain

containing L, such that $\{g_0, \zeta_{D_1,\gamma}\} = [g, \gamma]$. Since the Hausdorff distances between ∂L and B and between ∂L and ∂D_1 is less than η, the Hausdorff distance between ∂D_1 and B is less than 2η. So the triple $(\partial D_1, \overline{D}_1, g_1)$ is in the $\Phi, 2\eta$-neighborhood of (B, K, f).

Now we take a Jordan domain $D \subset\subset D_0 \cap D_1$ containing L such that the restriction of the curve γ to $[s_{D_1,\gamma}, s_{D_0,\gamma}]$ lies outside of $\overline{D}$. Let $t_1 = \sup\{t : \gamma(t) \in D\}$ and $\xi_1 = \gamma(t_1)$. Then the restriction γ_1 of the curve γ to $[t_1, s_{D_0,\gamma}]$ lies in $\overline{D}_1 \setminus D$. By Proposition 3.2 we can find a Φ, δ-approximation (D, h, ξ_1) of (L, g) which is also will lie in V. By Lemma 3.1 $\{h, \xi_1\} = \{f_0, \zeta_{D_0,\gamma}\}$. If we replace γ_1 with the restriction γ_2 of the curve γ to $[t_1, s_{D_1,\gamma}]$ the same reasoning shows that $\{h, \xi_1\} = \{g_0, \zeta_{D_1,\gamma}\}$. Thus $[f, \gamma] = [g, \gamma]$. □

In the future we will mostly use the space $\mathcal{S}^*(\mathbb{C}, W, M)$ and the following corollary, which is an immediate consequence of the preceding theorem, is rather useful.

Corollary 3.5. *Let (K_t, f_t) be a continuous curve in $\mathcal{S}^*(\mathbb{C}, W, M)$, $0 \le t \le 1$. Suppose that for all $t \in [0, 1]$ the point $\zeta_0 \in \partial K_t$, $f_t(\zeta_0) = w_0$ and γ is an access curve to K_t at $\gamma(0) = \zeta_0$. Then $[f_t, \gamma] = [f_0, \gamma]$ for all $t \in [0, 1]$.*

Let $f \in \mathcal{S}_{\zeta_0,w_0}(K, W, M)$ and let γ be an access curve to K at ζ_0. By Corollary 3.5 if $f \sim^h_\phi g$ then $[f, \gamma] = [g, \gamma]$. Hence the mapping $I_\gamma : [f]_{1,w_0} \to [f, \gamma]$ of $\mathcal{H}_{\zeta_0,w_0}[\partial K, K, W, M]$ into $\mathcal{H}_{1,w_0}[\mathbb{T}, \overline{\mathbb{D}}, W, M] = \eta_1(W, M, w_0)$ is well-defined. If $f \in \mathcal{S}_{1,w_0}(\overline{\mathbb{D}}, W, M)$ let $\iota(f)$ be the loop $f|_{\mathbb{T}}$ in W. Clearly, if $f \sim^h_{1,w_0} g$ in $\eta_1(W, M, w_0)$, then $\iota(f)$ and $\iota(g)$ are homotopic in $\pi_1(W, w_0)$. Hence the mapping

$$\iota_1 : \eta_1(W, M, w_0) \to \pi_1(W, w_0)$$

is also well-defined.

The role of an access curve γ is to choose where 1 is mapped by e_{D,ζ_0,ζ_1}. As the following example shows that this choice is important. Let $M = \mathbb{C}$, $W = \mathbb{C} \setminus (\{|\zeta + 4| \le 1\} \cup \{|\zeta - 4| \le 1\})$, $w_0 = 0$ and $\Pi(\zeta) = \zeta$. Let $K = \{|\zeta + 4| \le 2\} \cup [-2, 2] \cup \{|\zeta - 4| \le 2\})$ and $f : K \to M$ is defined as $f(\zeta) = \zeta$. Let $\zeta_0 = 0$, $\gamma_1(t) = it$ and $\gamma_2(t) = -it$, $t \ge 0$. Let D be a smooth Jordan domain containing K such that the Hausdorff distance between D and K is less than $\delta > 0$ and ∂D meets the imaginary axis in two points $i\sigma$ and $-i\sigma$, $\sigma > 0$. For small $\delta > 0$ we let $g_1(\zeta) = \zeta - i\sigma$ and $g_2(\zeta) = \zeta + i\sigma$. Then the pairs $(\overline{D}, g_1)$ and $(\overline{D}, g_2)$ can be as good approximations of (K, f) as we want. But the loops $g_1(e_{D,i\sigma}(e^{i\theta}))$ and $g_2(e_{D,-i\sigma}(e^{i\theta}))$, $0 \le \theta \le 2\pi$, are not equivalent in $\pi_1(W, w_0)$ and, consequently, $[g_1, \gamma_1] \ne [g_2, \gamma_2]$.

Two access curves γ_1 and γ_2 are *equivalent* if for every $\varepsilon > 0$ there is $\delta > 0$ such that if $0 < t_1, t_2 < \delta$ then the points $\gamma_1(t_1)$ and $\gamma_2(t_2)$ can be connected by a continuous curve α in $\mathbb{D}(\zeta_0, \varepsilon) \setminus K$. In the terminology of the prime ends theory (see [C]) it means that curves γ_1 and γ_2 determine the same prime end. In particular, if K is bounded by a Jordan curve (a homeomorphic image of a circle) then by a theorem of Carathéodory all access curves at any point of ∂K are equivalent.

The following result provides some information on the dependence of I_γ of γ.

Proposition 3.6. *Let $f \in \mathcal{S}_{\zeta_0,w_0}(K, W, M)$, $\zeta_0 \in \partial K$ and let γ be an access curve to K at ζ_0. Then:*

1. *if γ_0 and γ_1 are equivalent access curves then $I_{\gamma_0} = I_{\gamma_1}$;*
2. *if K is the closure of a Jordan domain then $[f, \gamma] = \{f, \zeta_0\}$.*

Proof. (1) We take δ from Lemma 3.1 and find Φ, δ-approximations (D_0, f_0) and (D_1, f_1) of (K, f) such that the diameter of γ_0 and γ_1 in D_0 and D_1 respectively is less than $\delta/3$ and $I_{\gamma_0}(f) = \{f_0, \zeta_{D_0,\gamma_0}\}$ and $I_{\gamma_1}(f) = \{f_1, \zeta_{D_1,\gamma_1}\}$. Then we connect $\gamma_0(t_0)$ and $\gamma_1(t_1)$ by a curve γ_2 in $\mathbb{D}(\zeta_0, \sigma) \subset\subset D_0 \cap D_1$ and with $\sigma < \delta/3$.

We take a Jordan domain $D \subset\subset D_0 \cap D_1$ such that $K \subset D$ and γ_2 does not meet $\overline{D}$ and let $t_2 = \sup\{t : \gamma_1(t) \in D\}$. Let (D, g) be a Φ, δ-approximation of (K, f) such that $g(\gamma_1(t_2)) = w_0$. By Lemma 3.1 $\{f_1, \zeta_{D_1,\gamma_1}\} = \{g, \gamma_1(t_2)\}$.

Let γ_3 be the curve which follows γ_0 from s_{D_0,γ_1} down to t_0, then γ_2 until it reaches γ_1 and then γ_1 down to $\gamma_1(t_2)$. The diameter of this curve is less than δ and it lies in $D_0 \setminus D$. Again by Lemma 3.1 $\{f_0, \zeta_{D_0,\gamma_0}\} = \{g, \gamma_1(t_2)\}$ and (1) is proved.

(2) is an immediate consequence of Lemma 3.1. □

The mapping I_γ need not to be surjective as the following result shows.

Theorem 3.7. *If $f \in \mathcal{S}_{\zeta_0,w_0}(K, W, M)$ and $f(K) \subset W$, then f is h-homotopic to the constant mapping $c \equiv w_0$. In particular, $\mathcal{H}_{\zeta_0,w_0}[\partial K, K, W, M]$ consists of one element if K has no interior.*

Proof. If $f \in \mathcal{S}_{\zeta_0,w_0}(K, W, M)$ is not equal to c then we assume, firstly, that f extends holomorphically to a neighborhood U of K. Since f maps K into W we can fix a Stein neighborhood V of the graph of f in $U \times W$. Then we choose a smooth Jordan domain D containing K such that f extends holomorphically to $\overline{D}$, f maps D into W, the graph Γ of f on $\overline{D}$ lies in V and $\rho(w_0, f(\gamma(t))) < \varepsilon$ when $0 \leq t \leq s_{D,\gamma}$ where the precise value of $\varepsilon > 0$ will be determined later.

Then we consider the continuous family ϕ_t, $0 \leq t < s_D$, of conformal automorphisms of D which move ζ_0 onto $\gamma(t)$ and leave $\zeta_{D,\gamma}$ in place. As the result the compact sets $K_t = \phi_t(K)$ will converge uniformly to $\zeta_{D,\gamma}$, the mappings $g_t = f \circ \phi_t$ will converge uniformly to the constant mapping $g \equiv f(\zeta_D)$ and $\rho(g_t(\zeta_0), w_0) < \varepsilon$.

Now if F is an imbedding of V into $\mathbb{C}^N$ and U is a neighborhood of $F(\Gamma)$ with the retraction P on $F(V)$, then we require $\varepsilon > 0$ to be so small that the mappings $\widetilde{f}_t(\zeta) = F(\zeta, g_t(\zeta)) - F(\zeta_0, g_t(\zeta_0)) + F(\zeta_0, w_0)$ map K into U. Then the continuous path $f_t = P_W \circ P \circ \widetilde{h}_t$ connects f and c.

For the general mapping f we note that by Theorem 4.3 in [P] it can be approximated by holomorphic mappings on neighborhoods of K whose restriction to K belongs to $\mathcal{S}_{\zeta_0,w_0}(K, W, M)$ and then the result follows from Corollary 2.6.

If K has no interior then any $f \in \mathcal{S}_{\zeta_0,w_0}(K, W, M)$ maps K into W and the result follows from the statement above. □

However, I_γ is surjective if K has a non-empty interior.

Theorem 3.8. *If K has a non-empty interior then the mapping I_γ is surjective.*

Proof. Suppose that $f \in \mathcal{S}_{1,w_0}(\overline{\mathbb{D}}, W, M)$. We want to show that there is $h \in \mathcal{S}_{\zeta_0,w_0}(K, W, M)$ such that $I_\gamma(h) = [f]_{1,w_0}$. The set $\hat{\Gamma} = \{(t, \zeta, \hat{f}(\zeta)), 0 \leq t \leq 1, \zeta \in \overline{\mathbb{D}}\}$ has a Stein neighborhood in $\mathbb{C} \times N \times M$. We take some extension of $\hat{f}$ to $\mathbb{C}$ and for some $\varepsilon > 0$ find $\delta > 0$ so we can use Lemma 2.5.

We take δ so small that f extends to $A = \{1 - \delta \leq |\zeta| \leq 1\}$ as a mapping in $\mathcal{A}(A, W)$. We choose a smooth Jordan domain D containing K and meeting γ. Mapping $D \setminus K$ onto an annulus $A_{r_1} = \{r_1 < |\zeta| < 1\}$ we introduce smooth Jordan

domains D_s whose boundaries a preimages of the circles of radius $r_1 < s \leq 1$ under this mapping.

Let ζ_1 be an interior point of K and let ϕ_s be a conformal mapping of D_s onto $\mathbb{D}$ moving ζ_1 into 0 and $\zeta_{D_s,\gamma}$ to 1. Let us show that for every $\delta > 0$ there is $r_1 < s_0 < 1$ such that for all $r_1 < s \leq s_0$ the mappings ϕ_s move ∂K into A. If not and there are a decreasing sequence $\{s_j\}$ converging to r_1 and a sequence $\{\xi_j\} \subset \partial K$ such that $|\phi_{s_j}(\xi_j)| < 1-\delta$ then we may assume that $\{\phi_{s_j}(\xi_j)\}$ converges to $\xi \in \overline{\mathbb{D}}(0, 1-\delta)$. By a theorem of Carathéodory (see [Go, Theorem II.5.1]) the sequence $\{\phi_{s_j}^{-1}\}$ converges uniformly on compacta to a conformal mapping ϕ of $\mathbb{D}$ onto the connected component of the interior of K containing ζ_1. But then

$$\phi(\xi) = \lim_{j\to\infty} \phi_{s_j}^{-1}(\phi_{s_j}(\xi_j)) = \lim_{j\to\infty} \xi_j \in \partial K$$

and this contradiction refutes this possibility.

Hence we may presume that $f(\phi_s(\gamma(t)))$ is defined when $r_1 < s \leq s_0$. Let us show that for any $\delta > 0$ there $r_1 < s_1 < s_0$ such that $\rho(f(\phi_s(\gamma(t))), w_0) < \delta$ when $0 \leq t \leq s_{D_s,\gamma}$ and $r_1 < s \leq s_1$. If there are a decreasing sequence $\{s_j\}$ converging to r_1 and a sequence of points $0 \leq t_j \leq s_{D_{s_j},\gamma}$ such that $\rho(f(\phi_{s_j}(\gamma(t_j)), w_0) \geq \delta$, then we denote by γ_j the restriction of γ to $[0, s_{D_{s_j},\gamma}]$. Then the harmonic measure of $\phi_{s_j}(\gamma_j)$ in $\mathbb{D}$ with respect to 0 is greater or equal to some $\varepsilon > 0$ while the harmonic measures of γ_j in D_{s_j} with respect to ζ_1 tend to 0 as $j \to \infty$. Since the harmonic measures are preserved by biholomorphisms, we see that the diameter of γ_j tends to 0 and $\rho(f(\phi_s(\gamma(t_j))), w_0) < \delta$ when j is large.

For $r_1 < s \leq s_1$ we set $L_s = \phi_{s_1}(\overline{D}_s)$ and let g_s to be the restrictions of f to L_s. We set $L_{r_1} = \phi_{s_1}(K)$ and g_{r_1} to be the restriction of f to L_{r_1}. Clearly (L_s, g_s) is a continuous path in $\mathcal{S}^*(\mathbb{C}, W, M)$. Let $\xi_s = \zeta_{D_s,\gamma}$. By Lemma 2.5 there is another continuous path (L_s, h_s) such that $h_s(\phi_{s_1}(\xi_s)) = w_0$. Moreover, $L_{s_1} = \overline{\mathbb{D}}$ and $h_{s_1} = f$.

By continuity the class $\{h_s, \xi_s\}$ stays constant when $r_1 < s \leq s_1$ and, therefore, is equal to $[f]_{1,w_0}$. If $p_s = h_s \circ \phi_{s_1}$ then we see that $\{p_s, \zeta_{D_s,\gamma}\} = [f]_{1,w_0}$ when $r_1 < s \leq s_1$. If $h = p_{r_1}$ then pairs $(\overline{D}_s, p_s)$ converge to (K, h) in $\mathcal{S}^*(\mathbb{C}, W, M)$ and we see that $I_\gamma(h) = [f]_{1,w_0}$. □

The following result allows us to modify compact sets and will be used as a major tool.

Lemma 3.9. *Suppose that K is the union of disjoint compact sets K_1 and K_2, which are connected and have connected complements, and a simple curve $\alpha : [0,1] \to \mathbb{C}$, which connects K_1 and K_2 and $\alpha \cap K_1 = \{b = \alpha(1)\}$, $\alpha \cap K_2 = \{a = \alpha(0)\}$. Let γ be an access curve to K at a and let $f \in \mathcal{S}_{a,w_0}(K, W, M)$, $f(b) = w_1$. There is a connected compact set L consisting of K_2, a curve $\beta(t) = \alpha(t)$, $0 \leq t \leq t_0 \leq 1$, and a closed disk D attached to β at $\beta(t_0)$ and $g \in \mathcal{S}_{a,w_0}(L, W, M)$ such that $g(\beta(t_0)) = w_1$, $[f, \gamma] = [g, \gamma]$, $[g|_{\beta\cup D}, \gamma] = [f|_{\alpha\cup K_1}, \gamma]$ and $\{g|_D, \beta(t_0)\} = [f|_{K_1}, \alpha]$.*

Proof. By the definition of $[f|_{K_1}, \alpha]$ the pair $(K_1, f|_{K_1})$ has a Φ, ε-approximation (Ω, h), where Ω is a smooth Jordan domain containing K_1, such that $\overline{\Omega}$ does not meet K_2, $h(\zeta_{\Omega,\alpha}) = w_1$ and $\{h, \zeta_{\Omega,\alpha}\} = [f|_{K_1}, \alpha]$, where $\zeta_{\Omega,\alpha} = \alpha(t_0)$ and $t_0 = \min\{t : \alpha(t) \in \partial\Omega\}$. Let η be taken from Theorem 3.4. We may assume that ε is so small that we can extend h continuously to the curve β so that $\rho(h(t), f(t)) < \eta$ and $h(a) = w_0$ and $(\beta \cup \overline{\Omega}, h)$ lies in the Φ, η-neighborhood of $(\alpha \cup K_1, f_{\alpha\cup K_1})$. If

we extend h to K_2 as f, then the pair $(K_2 \cup \beta \cup \overline{\Omega}, h)$ is in the Φ, η-neighborhood of (K, f). By Theorem 3.4 $[f, \gamma] = [h, \gamma]$, $[h|_{\beta \cup \overline{\Omega}}, \gamma] = [f|_{\alpha \cup K_1}, \gamma]$ and $[h|_{\beta \cup \overline{\Omega}}, \gamma] = [f|_{\alpha \cup K_1}, \gamma]$.

Now we take a disk $D \subset \Omega$ such that $\zeta_{\Omega,\alpha} \in \partial D$ but $\overline{D} \subset \Omega \cup \{\zeta_{\Omega,\alpha}\}$. The set $\overline{\Omega} \setminus D$ is conformally equivalent to the strip $\{0 \leq \mathbf{Im}\, \zeta \leq 1\}$ and we let Ω_t to be simply connected domains in Ω whose boundaries, except $\zeta_{\Omega,\alpha}$, are moved to lines $\{\mathbf{Im}\, \zeta = t\}$, $0 < t < 1$, by this equivalence so that $\partial\Omega$ goes to $\{\mathbf{Im}\, \zeta = 0\}$. Clearly we get a Radó continuous family of simply connected domains Ω_t. Let K^t be compact sets consisting of $\overline{\Omega}_t$, K_2 and the curve β. Let ϕ_t be a continuous family of conformal mappings of $\overline{\Omega}_t$ onto $\overline{\Omega}$ such that $\phi_0(\zeta) \equiv \zeta$ and $\phi_t(\zeta_{\Omega,\alpha}) = \zeta_{\Omega,\alpha}$. Define $f^t \in \mathcal{S}_{a,w_0}(K^t, W, M)$ as $f^t(\zeta) = h(\phi_t(\zeta))$ on $\overline{\Omega}_t$ and f on K_2 and β. Then by Corollary 3.5 $[f^t, \gamma] = [h, \gamma]$, $[h|_{\beta \cup D}, \gamma] = [f^t|_{\alpha \cup K_1}, \gamma]$ and $\{f^t|_{\overline{\Omega}_t}, \zeta_{\Omega,\alpha}\} = \{h, \zeta_{\Omega,\alpha}\}$ for all $t \in [0, 1]$.

The pair of the set $L = K^1$ consisting of $\overline{D}$, K_2 and β and the mapping $g = f^1$ satisfies all requirements of the Lemma. □

Remark: If α is a smooth curve then by Corollary 3.5 we can shift D along α so it becomes attached to b.

4. Holomorphic fundamental semigroup of Riemann domains

If $f \in \mathcal{S}_{1,w_0}(\overline{\mathbb{D}}, W, M)$ we will denote $[f]_{1,w_0}$ by $[f]$. To introduce on $\eta_1(W, M, w_0)$ a semigroup structure compatible with ι_1 we need additional construction since in the standard definition the sum of two loops cannot be realized as a boundary of an analytic disk.

Suppose that $f_1, f_2 \in \mathcal{S}_{1,w_0}(\overline{\mathbb{D}}, W, M)$ are representatives of equivalence classes $[f_1]$ and $[f_2]$ respectively in $\eta_1(W, M, w_0)$. Let $K \subset \mathbb{C}$ be the union of $K_1 = \{|\zeta - 1| \leq 1\}$ and $K_2 = \{|\zeta + 1| \leq 1\}$ and let $\gamma(t) = -it$, $0 \leq t \leq 1$. Then γ is an access curve for K to 0. We define the mapping

$$h_{f_1,f_2}(\zeta) = \begin{cases} f_1(1 - \zeta),\ \zeta \in \partial K_1, \\ f_2(1 + \zeta),\ \zeta \in \partial K_2 \end{cases}$$

of ∂K into W. The mapping $\hat{h}_{f_1,f_2}$ maps K into M so $h_{f_1,f_2} \in \mathcal{S}_{0,w_0}(K, W, M)$.

We let $[f_1] \star [f_2] = I_\gamma(h_{f_1,f_2})$. If f_1 and f_2 are h-homotopic to g_1 and g_2 respectively in $\mathcal{S}_{1,w_0}(\overline{\mathbb{D}}, W, M)$, then evidently h_{f_1,f_2} is h-homotopic to h_{g_1,g_2} in $\mathcal{S}_{0,w_0}(K, W, M)$. Hence the class $[f_1] \star [f_2]$ is well defined.

One of the advantages of the $\star$ operation is its help to calculate the homotopic type of holomorphic mappings of compact sets. Let α_1 and α_2 be two simple curves connecting the origin and points ζ_1 and ζ_2 in $\mathbb{C}$ and meeting only at the origin. We attach to these points two disjoint compact sets K_1 and K_2, which are connected and have connected complements, so that $K_j \cap \alpha_j = \{\zeta_j\}$, $j = 1, 2$, and $K_j \cap \alpha_k = \emptyset$, $j \neq k$. Let $L_1 = \alpha_1 \cup K_1$ and $L_2 = \alpha_2 \cup K_2$ and $L = L_1 \cup L_2$. Let γ be an access curve to L at the origin such that if we move by γ and then by ∂L counterclockwise, then we meet L_1 first and then L_2.

Proposition 4.1. *Suppose that $f \in \mathcal{S}_{0,w_0}(L, W, M)$. Let f_j be the restriction of f to L_j, $j = 1, 2$. Then $I_\gamma(f) = I_\gamma(f_1) \star I_\gamma(f_2)$.*

Proof. Deforming curves α_1 and α_2 and the mapping f near the origin we may assume that there is $t_0 > 0$ such that $\alpha_1(t) = t$ and $\alpha_2(t) = -t$ and $f(\alpha_1(t)) =$

$f(\alpha_2(t)) = w_0$ when $0 \le t \le t_0$. To use Lemma 3.9 we split L into compact sets K_1' which consists of the restriction of the curve α_1 to $[t_0, 1]$ and K_1 while K_2' consists of α_2 and K_2. The role of the connecting curve α is played by the restriction of the curve α_1 to $[0, t_0]$. The same splitting is applied to L_1 but in this case K_2' is just the origin.

By Lemma 3.9 we can replace in both cases K_1' by a closed disk attached to $\alpha_1(t_0)$ and the mapping f by a mapping g so that $I_\gamma(f) = I_\gamma(g)$ and $I_\gamma(f_1) = I_\gamma(g|_{K_1' \cup \alpha})$.

We repeat the same trick for K_2 and obtain a compact set A consisting of the intervals $I_1 = [0, t_0]$ and $I_2 = [-t_0, 0]$ and closed disks D_1 and D_2 attached to t_0 and $-t_0$ respectively. Then we construct a continuous path in $\mathcal{S}^*(\mathbb{C}, W, M)$. At the first step we rotate the disks D_1 and D_2 around t_0 and $-t_0$ respectively so that the intervals I_1 and I_2 become perpendicular to the boundary of the disks. The mappings g_t are defined by compositions with these rotations. Since this is a continuous path in $\mathcal{S}^*(\mathbb{C}, W, M)$ by Corollary 3.5 homotopic types do not change.

Then we shrink intervals I_1 and I_2 to the origin applying dilations $d_t(\zeta) = t\zeta$, $0 \le t \le 1$, with the simultaneous parallel translations of the disks D_1 and D_2 along the real line so they stay connected with the intervals. Again the mappings g_t are defined on the disks by compositions with these translations and stay equal to w_0 on the interval.

Finally, we dilate the disks using the mappings $t\zeta$ to make them of radius 1. In this way we obtained a compact set K consisting of two disks exactly as in the definition of the $\star$ operation with mappings g_1 and g_2 on the disks. Let $g = h_{g_1, g_2}$. By construction $I_\gamma(g_1) = I_\gamma(f_1)$, $I_\gamma(g_2) = I_\gamma(f_2)$ and $I_\gamma(g) = I_\gamma(f)$. Since $I_\gamma(g_1) = [g_1]$, $I_\gamma(g_2) = [g_2]$ and $I_\gamma(g) = [g_1] \star [g_2]$ the result follows. $\square$

This construction allows us to prove that $\eta_1(W, M, w_0)$ with the operation $\star$ is a semigroup.

Theorem 4.2. *The operation $\star$ induces on $\eta_1(W, M, w_0)$ the structure of a semigroup with unity.*

Proof. The unity is the class of the constant mapping equal to w_0 on $\mathbb{T}$. If, say, $f_1 \equiv w_0$ then continuously shrinking K_1 to the origin leaving the functions equal to w_0 we will get a continuous path in $\mathcal{S}^*(\mathbb{C}, W, M)$ which ends at $(K_2, f_2(1 + \zeta))$. By Corollary 3.5 $I_\gamma(h_{f_1, f_2}) = [f_2]$.

To prove that the operation $\star$ is associative we consider a compact set L consisting of three intervals $I_1 = [0, 1]$, $I_2 = [0, i]$, $I_3 = [-1, 0]$ and three closed disks $D_1 = \{|\zeta - 2| \le 1\}$, $D_2 = \{|\zeta - 2i| \le 1\}$ and $D_3 = \{|\zeta + 2| \le 1\}$. The access curve $\gamma = [-i, 0]$. Given $f_1, f_2, f_3 \in \mathcal{S}_{1, w_0}(\overline{\mathbb{D}}, W, M)$ we define the mapping f on L to be equal to w_0 on intervals I_1, I_2, I_3 and $f_1(2 - \zeta)$ on D_1, $f_2(2 + i\zeta)$ on D_2 and $f_3(2 + \zeta)$ on D_3.

Let f_j' be the restriction of f to $I_j \cup D_j$, $j = 1, 2, 3$. Shifting continuously D_j to the origin we see that $I_\gamma(f_j') = [f_j]$. Let f_j'' be the restriction of f to $\cup_{k \ne j} I_k \cup D_k$. By Proposition 4.1 $I_\gamma(f_3'') = I_\gamma(f_1') \star I_\gamma(f_2') = [f_1] \star [f_2]$. Also by Proposition 4.1 $I_\gamma(f) = I_\gamma(f_3'') \star I_\gamma(f_3') = ([f_1] \star [f_2]) \star [f_3]$. Now if repeat this argument taking instead f_1'' then we get $I_\gamma(f) = I_\gamma(f_1') \star I_\gamma(f_1'') = [f_1] \star ([f_2] \star [f_3])$. $\square$

Another useful tool to calculate homotopic types using the $\star$ operation is the content of the following proposition. Suppose that D is a Jordan domain in $\mathbb{C}$ and $\zeta_0 \in \partial D$. Suppose that there are k simple curves $\alpha_1, \dots, \alpha_k$ in $\overline{D}$ such that the

curves meet ∂D only at their endpoints, endpoints of each curve are not equal and the curves may meet each other only at endpoints. The curves are numbered in such a way that if we move counterclockwise from ζ_0 along ∂D until we reach an endpoint of one of these curves such that the domain between this curve and ∂D contains no other curves, then this curve is α_1. We denote the domain between α_1 and ∂D by D_1. We also denote by ζ_1 the endpoint of α_1 we encountered first. Let K_1 be the compact set consisting of $\overline{D}_1$ and the arc in ∂D connecting ζ_0 and ζ_1. The complement of D_1 in D is also a Jordan domain and if we repeat this process in D_1 then we get α_2, D_2 and K_2 and so on. Thus we obtain Jordan domains D_j and compact sets K_j, $1 \le j \le k+1$, and we will say that the curves $\alpha_1, \dots, \alpha_k$ *divide* D.

Let $B = \partial D \cup \alpha_1 \cup \cdots \cup \alpha_k$ and $f \in \mathcal{S}_{\zeta_0,w_0}(B, \overline{D}, W, M)$. Let f_j be the restriction of f to ∂K_j. Then $f_j \in \mathcal{S}_{\zeta_0,w_0}(K_j, W, M)$.

Proposition 4.3. *Suppose that D is a Jordan domain in $\mathbb{C}$, $\zeta_0 \in \partial D$ and γ is an access curve to D at ζ_0. Let $\alpha_1, \dots, \alpha_k$ be simple continuous curves in $\overline{D}$ dividing D into domains D_j, $1 \le j \le k+1$. If $f \in \mathcal{S}_{\zeta_0,w_0}(B, \overline{D}, W, M)$ then*

$$I_\gamma(f) = I_\gamma(f_1) \star \cdots \star I_\gamma(f_{k+1}).$$

Proof. Since the mappings f_j are defined inductively it suffices to prove this proposition for $k = 1$. The rest follows by induction.

At the first step we will separate D_1 and D_2 along α_1. Let $\alpha_1 : [0,1] \to \overline{D}$, $\alpha_1(0) = \zeta_1$ and $\alpha_1(1) = \xi_1$. We fix conformal mappings Ψ_1 and Ψ_2 of $\overline{\mathbb{D}}$ onto $\overline{D}_1$ and $\overline{D}_2$ respectively such $\Psi_1(1) = \Psi_2(1) = \zeta_1$ and $\Psi_1(-1) = \Psi_2(-1) = \xi_1$. We may extend f holomorphically into a neighborhood V of α_1 as a mapping into W. There is $t_0 > 0$ such that all arcs β_t of unit circles centered at it, $0 \le t \le t_0$, lying in $\mathbb{D}$ belong to $\Psi_1^{-1}(V)$. Let E_t be the closed set in $\overline{\mathbb{D}}$ bounded by β_t and $\mathbb{T}$ and containing 0. We let K_1^t be the union of $\Psi_1(E_t)$ and the arc in ∂D connecting ζ_0 and ζ_t, where $\zeta_t = \Psi_1(\xi_t)$ and ξ_t is the point where β_t meets $\mathbb{T}$ and $\mathbf{Re}\,\xi_t > 0$. Set f_1^t to be the restriction of f to ∂K_1^t. Then (K_1^t, f_1^t) is a continuous path in $\mathcal{S}^*(\mathbb{C}, W, M)$ and $f_t(\zeta_0) = w_0$. By Corollary 3.5 $I_\gamma(f_1^t) = I_\gamma(f_1)$ when $0 \le t \le t_0$.

We do the same with D_2 getting a continuous path (K_2^t, f_2^t) in $\mathcal{S}^*(\mathbb{C}, W, M)$ with $f_2^t(\zeta_0) = w_0$ such that $I_\gamma(f_2^t) = I_\gamma(f_2)$ when $0 \le t \le t_0$.

Let K^t be the union of K_1^t and K_2^t and let f^t be the restriction of f to ∂K^t. If η is taken from Theorem 3.4 then the triple $(\partial K^t, K^t, f^t)$ lies in the Φ, η-neighborhood of (B, K, f) when t is small and by this theorem $I_\gamma(f^t) = I_\gamma(f)$.

In the last step we separate in the same way K_2^t from K_1^t along the arc in ∂D connecting ζ_0 and ζ_t. Again homotopic types will not change. We apply Proposition 4.1 to show that $I_\gamma(f) = I_\gamma(f_1) \star I_\gamma(f_2)$. □

5. Examples of holomorphic fundamental semigroups

In this section $W = A_{s,r} = \{s < |z| < r\}$, where $0 < s < 1 < r$, and $M = \mathbb{CP}^1$ or $M = \mathbb{D}(0, x) = \{|\zeta| < x\}$, where $r \le x \le \infty$. We fix $\Pi(z) = z$ and $w_0 = 1$.

The examples below show that the mapping $\iota_1 : \eta_1(W, M, w_0) \to \pi_1(W, w_0)$ need not to be injective or surjective.

Theorem 5.1. *The semigroup $\eta_1(A_{s,r}, \mathbb{CP}^1, w_0)$ is isomorphic to $\mathbb{N}_0 \oplus \mathbb{N}_0$, where $\mathbb{N}_0$ is the semigroup by addition of non-negative integers. Under this isomorphism the class of $f \in \mathcal{S}_{1,w_0}(\overline{\mathbb{D}}, A_{s,r}, \mathbb{CP}^1)$ is mapped into (m, n), where m and n are the numbers of zeros and poles of $\hat{f}$ respectively counted with multiplicity.*

The semigroup $\eta_1(A_{s,r}, \mathbb{D}(0,x), w_0)$ is isomorphic to $\mathbb{N}_0$. Under this isomorphism the class of $f \in \mathcal{S}_{1,w_0}(\overline{\mathbb{D}}, A_{s,r}, \mathbb{D}(0,x))$ is mapped into m, where m is the number of zeros of $\hat{f}$ counted with multiplicity.

Proof. Firstly, we show that if $f, g \in \mathcal{S}_{1,w_0}(\overline{\mathbb{D}}, A_{s,r}, \mathbb{CP}^1)$ and $\hat{f}$ and $\hat{g}$ have the same numbers of zeros and poles, then $[f] = [g]$. For this we define $F(z,\zeta) = \alpha(z)(\zeta - z)(1 - \bar{z}\zeta)^{-1}$, where $\alpha(z) = (1-z)^{-1}(1-\bar{z})$, so $F(z,1) = 1$. If a_k, $1 \le k \le m$, are zeros of f and b_j, $1 \le j \le n$, are poles of f, then we can write $\hat{f}(\zeta) = Z(\zeta)(P(\zeta))^{-1}h(\zeta)$, where $Z(\zeta) = \prod_{j=1}^m F(a_j,\zeta)$, $P(\zeta) = \prod_{j=1}^n F(b_j,\zeta)$ and $h \in \mathcal{S}_{1,w_0}(\overline{\mathbb{D}}, A_{s,r}, \mathbb{CP}^1)$ has no zeros or poles. Since $s < |h(\zeta)| < r$ on $\mathbb{T}$, we see that $s < |h(\zeta)| < r$ on $\overline{\mathbb{D}}$.

Take two distinct points $a, b \in \mathbb{D}$. Let $\alpha_k(t)$, $1 \le k \le m$, and $\beta_j(t)$, $1 \le j \le n$, be families of continuous curves on $[0,1]$ in $\mathbb{D}$ such that $\alpha_k(0) = a_k$, $\alpha_k(1) = a$, $\beta_j(0) = b_j$, $\beta_j(1) = b$, and no two curves intersects each other except at end points. For $0 \le t \le 1$ let $Z_t(\zeta) = \prod_{j=1}^m F(\alpha_j(t),\zeta)$ and $P_t(\zeta) = \prod_{j=1}^n F(\beta_j(t),\zeta)$. Define $h_t(\zeta) = h((1-t)\zeta + t)$ and $f_t = Z_t P_t h_t$. Then $f_t(1) = 1 = w_0$, $f_0 = f$ and $f_1(\zeta) = F^m(a,\zeta)F^{-n}(b,\zeta)$. Thus we see that mappings with the same numbers of zeros and poles are h-homotopic to the same mapping and, hence, their homotopic types are the same.

If f_t is a continuous path in $\mathcal{S}(\overline{\mathbb{D}}, A_{s,r}, \mathbb{CP}^1)$, then the mappings $\hat{f}_t$ form a continuous path in $\mathcal{A}(\overline{\mathbb{D}}, \mathbb{CP}^1)$. Hence the number of zeros and poles counted with multiplicity stays constant. Therefore the mapping $R([f]) = (m,n)$ of $\eta_1(W, \mathbb{CP}^1, w_0)$ into $\mathbb{N}_0 \oplus \mathbb{N}_0$, where m and n are the numbers of zeros and poles of $\hat{f}$ respectively counted with multiplicity, is well defined. By the argument above the mapping R is injective and, evidently, it is surjective. Let us show that if f_1 has m_1 zeros and n_1 poles and f_2 has m_2 zeros and n_2 poles, then $R([f_1] \star [f_2]) = (m_1 + m_2, n_1 + n_2)$. This follows immediately from the definition of the $\star$ operation because a sufficiently good Φ, ε-approximations of the mapping h_{f_1,f_2} in the definition has exactly $m_1 + m_2$ zeros and $n_1 + n_2$ poles.

The case when $M = \mathbb{D}(0,x)$ follows from this argument if we take into account that the number of poles is equal to 0. $\square$

Another example when ι_1 is not be injective is based on an example by Wermer ([W1, W2]). He constructed a strongly pseudoconvex domain $\Omega \subset \mathbb{C}^3$ diffeomorphic to a ball and a holomorphic imbedding $F(z,w,t) = (z, zw + t, zw^2 - w + 2tw)$ of Ω into $\mathbb{C}^3$ such that the mapping $f(\zeta) = (\zeta, 1, 0) \in \mathcal{S}(\overline{\mathbb{D}}, F(\Omega), \mathbb{C}^3)$ but $f(0) \notin F(\Omega)$. While $\iota_1([f]) = 0$ the mapping f is not h-homotopic to a constant mapping because it will contradict the continuity principle. However, it was proved by Nemirovskii in [Ne] that if Ω is a strongly pseudoconvex domain in $\mathbb{C}^2$ diffeomorphic to the ball then $\eta_1(\Omega, \mathbb{C}^2, w_0) = 0$.

6. Properties of holomorphic fundamental semigroups

Let (W_1, Π_1) and (W_2, Π_2) be two Riemann domains over two complex manifolds M_1 and M_2 respectively. Suppose $w_1 \in W_1$, $w_2 \in W_2$ and there are holomorphic mappings $\phi : W_1 \to W_2$ such that $\phi(w_1) = w_2$ and $\psi : M_1 \to M_2$ which satisfy $\psi \circ \Pi_1 = \Pi_2 \circ \phi$. Then for any $f \in \mathcal{S}(K, W_1, M_1)$ we have $\Pi_2 \circ \phi \circ f = \psi \circ \Pi_1 \circ f = \psi \circ \hat{f}$. So $\widehat{\phi \circ f} = \psi \circ \hat{f}$ and we get a continuous mapping from $\mathcal{S}^*(N, W_1, M_1)$ to $\mathcal{S}^*(N, W_2, M_2)$ which maps a pair (K, f) to $(K, \phi \circ f)$. Hence, firstly, the mapping

from $\mathcal{S}_{1,w_1}(\overline{\mathbb{D}}, W_1, M_1)$ to $\mathcal{S}_{1,w_2}(\overline{\mathbb{D}}, W_2, M_2)$ induces a well defined mapping ϕ_* from $\eta_1(W_1, M_1, w_1)$ to $\eta_1(W_2, M_2, w_2)$ given by $\phi_*([f]) = [\phi \circ f]$. Secondly, if γ is an access curve to K, then $\phi_*([f, \gamma]) = [\phi \circ f, \gamma]$. In particular, if (K, h_{f_1,f_2}) is the pair in the definition of the $\star$ operation, then

$$\begin{aligned}&\phi_*([f_1] \star [f_2]) = \phi_*([h_{f_1,f_2}, \gamma])\\&= [\phi \circ h_{f_1,f_2}, \gamma] = [h_{\phi\circ f_1,\phi\circ f_2}, \gamma] = [\phi \circ f_1] \star [\phi \circ f_2] = \phi_*[f_1] \star \phi_*[f_2].\end{aligned}$$

This leads us to the following proposition.

Proposition 6.1. *The induced mapping $\phi_* : \eta_1(W_1, M_1, w_1) \to \eta_1(W_2, M_2, w_2)$ is a homomorphism.*

This proposition has the following corollary.

Corollary 6.2. *Let $f \in \mathcal{S}_{1,w_2}(\overline{\mathbb{D}}, W_2, M_2)$ and let $[f]^{\star k}$ be the product of k classes $[f]$. Then $[f]^{\star k} = [f(\zeta^k)]$.*

Proof. We may assume that $\hat{f}$ is defined on $\mathbb{D}(0, r)$, $r > 1$, and f maps $A_{r^{-1},r}$ into W_2. Set $W_1 = A_{r^{-1},r}$, $M_1 = \mathbb{D}(0, r)$ and $w_1 = 1$. Let $\phi = f$ and $\psi = \hat{f}$. By Proposition 6.1 and Theorem 5.1 we have

$$[f(\zeta^k)] = \phi_*([\zeta^k]) = \phi_*([\zeta]^{\star k}) = \phi_*([\zeta])^{\star k} = [f]^{\star k}.$$

□

Let (W_1, Π_1) and (W_2, Π_2) be two Riemann domains over two complex manifolds M_1 and M_2 respectively. Then clearly $(W_1 \times W_2, (\Pi_1, \Pi_2))$ is a Riemann domain over $M_1 \times M_2$.

Theorem 6.3. *If (W_1, Π_1) and (W_2, Π_2) are two Riemann domains over two complex manifolds M_1 and M_2 respectively, then*

$$\eta_1(W_1 \times W_2, M_1 \times M_2, (w_1, w_2)) \cong \eta_1(W_1, M_1, w_1) \times \eta_1(W_2, M_2, w_2).$$

Proof. Let $p_i : W_1 \times W_2 \to W_i$ and $q_i : M_1 \times M_2 \to M_i$ be projection maps for $i = 1, 2$. Then by Proposition 6.1 induced mappings p_{i*} from $\eta_1(W_1 \times W_2, M_1 \times M_2, (w_1, w_2))$ into $\eta_1(W_i, M_i, w_i)$ given by $p_{i*}([(f_1, f_2)]) = [f_i]$ are homomorphisms. Now define a mapping

$$\phi : \eta_1(W_1 \times W_2, M_1 \times M_2, (w_1, w_2)) \to \eta_1(W_1, M_1, w_1) \times \eta_1(W_2, M_2, w_2)$$

by taking $\phi = (p_{1*}, p_{2*})$, i.e.

$$\phi([(f_1, f_2)]) = (p_{1*}([(f_1, f_2)]), p_{2*}([(f_1, f_2)])) = ([f_1], [f_2]).$$

Since p_{1*} and p_{2*} are homomorphisms clearly ϕ is a homomorphism.

To show that ϕ is an isomorphism we construct its inverse. To do that define the mapping

$$\psi : \eta_1(W_1, M_1, w_1) \times \eta_1(W_2, M_2, w_2) \to \eta_1(W_1 \times W_2, M_1 \times M_2, (w_1, w_2))$$

by taking $\psi(([f_1], [f_2])) = [(f_1, f_2)]$. This mapping ψ is well defined since for any two continuous paths f_t in $\mathcal{S}_{1,w_1}(\overline{\mathbb{D}}, W_1, M_1)$ and g_t in $\mathcal{S}_{1,w_2}(\overline{\mathbb{D}}, W_2, M_2)$, (f_t, g_t) is a continuous path in $\mathcal{S}_{1,(w_1,w_2)}(\overline{\mathbb{D}}, W_1 \times W_2, M_1 \times M_2)$. It is easy to see that ϕ and ψ are inverses of each other. Hence ϕ is an isomorphism. □

We want to show that the holomorphic fundamental semigroup does not depend on the choice of base points. Let w_0 and w_1 be two points in W. Let $\alpha(t)$, $t \in [0,1]$, be a continuous curve in W with $\alpha(0) = w_0$ and $\alpha(1) = w_1$. Let L be a compact set on the plane consisting of the interval $I = [0,1]$ and the closed disk $D = \{|\zeta - 2| \le 1\}$. Given a mapping $f \in \mathcal{S}_{1,w_1}(\overline{\mathbb{D}}, W, M)$ we define a mapping $\widetilde{f}$ on L to be equal to α on I and to $f(2-\zeta)$ on ∂D. Clearly, $\widetilde{f} \in \mathcal{S}_{0,w_0}(L, W, M)$.

We take the access curve $\gamma(t) = -it$, $0 \le t \le 1$, to L at the origin. Clearly, if $[f] = [g]$, then $[\widetilde{f}, \gamma] = [\widetilde{g}, \gamma]$. Hence we have a well-defined mapping F_α from $\eta_1(W, M, w_1)$ into $\eta_1(W, M, w_0)$.

First of all, by Corollary 3.5 any curve connecting w_0 to w_1 which is homotopic to α will give us the same mapping F_α. Thus F_α depends only on the homotopic type $\{\alpha\}$ of α in $\pi_1(W, w_0, w_1)$. Secondly, we let α^{-1} to be the curve $(\alpha^{-1})(t) = \alpha(1-t)$ for $0 \le t \le 1$ and denote by $\alpha\beta$ the curve on $[0,1]$ defined as $\alpha(2t)$ when $0 \le t \le 1/2$ and as $\beta(2t-1)$ when $1/2 \le t \le 1$. Then $\iota_1(F_\alpha([f]))$ is equal to the homotopic type of $\alpha f \alpha^{-1}$ in $\pi_1(W, w_0)$. Slightly abusing the notation we will denote also by F_α the homomorphism $\alpha\beta\alpha^{-1}$ mapping $\pi_1(W, w_1)$ into $\pi_1(W, w_0)$. And, thirdly, if $g \in \mathcal{S}_{1,w_0}(\overline{\mathbb{D}}, W, M)$ then the mapping $F_g = F_\alpha$, where $\alpha(t) = g(e^{2\pi i t})$.

Proposition 6.4. *Let $\{w_0, w_1, w_2\} \subset W$, α be a curve which connects w_0 and w_1, β be a curve which connects w_1 and w_2, $f_1, f_2 \in \mathcal{S}_{1,w_1}(\overline{\mathbb{D}}, W, M)$ and $f, g \in \mathcal{S}_{1,w_0}(\overline{\mathbb{D}}, W, M)$. Then,*

(1) $F_{\alpha\beta} = F_\alpha \circ F_\beta$.
(2) $F_{\alpha^{-1}} \circ F_\alpha([f_1]) = [f_1]$.
(3) $F_\alpha([f_1] \star [f_2]) = F_\alpha([f_1]) \star F_\alpha([f_2])$.
(4) $F_g([f]) \star [g] = [g] \star [f]$.
(5) $F_g([g]) = [g]$.

Proof. (1) Given $f \in \mathcal{S}_{1,w_2}(\overline{\mathbb{D}}, W, M)$ we consider the pair $(L, \widetilde{f})$ as above for the curve $\alpha\beta$. By Corollary 3.5 we may dilate $[0,1]$ so that L consists of intervals $I_1 = [0,1]$, $I_2 = [1,2]$ and the closed disk D of radius 1 attached to 2 and $\widetilde{f}|_{I_1} = \alpha$ and $\widetilde{f}|_{I_2} = \beta$. By Lemma 3.9 we can replace L with a compact set L' which consists of I_1 and a disk D' attached to 1 and replace $\widetilde{f}$ with g on L' so that $[g, \gamma] = [f, \gamma]$ and $\{g|_{D'}, 1\} = [\widetilde{f}|_{I_2 \cup D}, I_1] = F_\beta([f])$. Now $F_{\alpha\beta}([f]) = F_\alpha(F_\beta([f]))$ by the definition of F_α.

(2) follows from (1) because $\alpha^{-1}\alpha$ and $\alpha\alpha^{-1}$ are homotopic to constant curves.

(3) Consider a compact set K consisting of disks $K_1 = \{|\zeta - 1| \le 1\}$ and $K_2 = \{|\zeta + 1| \le 1\}$ and the interval $I = [-i, 0]$. We define a mapping f on $K_1 \cup K_2$ as in the definition of the $\star$ operation and let $f(-it) = \alpha(t)$ when $0 \le t \le 1$. Let $\gamma = [-2i, -i]$ be an access curve to K at $-i$.

By Lemma 3.9 we can replace $K_1 \cup K_2$ with a closed disk D attached to the origin and f_1 and f_2 with $g \in \mathcal{S}_{0,w_0}(\overline{D}, W, M)$ so that $\{g, 0\} = [f_1] \star [f_2]$. Thus $I_\gamma(f) = F_\alpha([f_1] \star [f_2])$.

On the other hand let K^t be the compact sets consisting of disks $K_1^t = \{|\zeta - 1 - t| \le 1\}$ and $K_2^t = \{|\zeta + 1 + t| \le 1\}$, intervals $I_1^t = [t, -i + t]$, $I_2^t = [-t, -t - i]$ and the interval $I^t = [-t - i, -i + t]$. We define a mapping f^t on K_1 as $f_1^t(1 + t - \zeta)$ and on K_2 as $f_2^t(\zeta + 1 - t)$. We let $f^t(t - si) = \alpha(s)$ and $f^t(-t - si) = \alpha(s)$ when $0 \le s \le 1$ and let $f \equiv w_0$ on I^t. The path (K^t, f^t) is continuous when $0 \le t \le 1$ and by Corollary 3.5 $I_\gamma(f^t) = I_\gamma(f)$. Since $K^0 = K$ and $f^0 = f$ we

see that $I_\gamma(f^1) = I_\gamma(f)$. Now we apply Proposition 4.1 where $K_1 := K_1^1 \cup I_1^1$, $K_2 := K_2^1 \cup I_1^1$ and $\alpha := I^1$ to see that $I_\gamma(f) = F_\alpha([f_1]) \star F_\alpha([f_2])$.

(4) Consider a compact set K consisting of the disks $D^1 = \{|\zeta| \leq 1\}$ and $D^2 = \{|\zeta - 3| \leq 1\}$ and the interval $I = [1, 2]$. We define a mapping $h(\zeta)$ on D^1 as $g(\zeta)$ and on D^2 as $f(3-\zeta)$. Let $h(t) = g(e^{2\pi i(t-1)})$ on I. Let $\gamma = [-i+1, 1]$ be an access curve to K at 1. Then $[h, \gamma] = F_g([f]) \star [g]$.

Consider the continuous family of compact sets K_s, $0 \leq s \leq 1/2$, consisting of the disk D^1, an interval $I_s = [e^{2\pi si}, (2-s)e^{2\pi si}]$ and the closed disk D_s^2 attached to $(2-s)e^{2\pi si}$. The mapping h_s on K_s is defined as h on D^1 and as $h((s + |\zeta| - 1)$ when $\zeta \in I_s$. The mapping h_s on D_s^2 is defined as a composition of h on D^2 and a conformal mapping that maps D_s^2 onto D^2 moving $(2-s)e^{2\pi si}$. Simply speaking we rotate $I \cup D^2$ around D^1 leaving one end of I_s attached normally to D^1. Clearly, the pairs (K_s, h_s) form a continuous path and $[h_s, \gamma] = F_g([f]) \star [g]$.

When $s = 1/2$ the set $K_{1/2}$ consists of D^1, $I_{1/2} = [-1, -3/2]$ and the disk $D_{1/2}^2$. Since all access curves to $K_{1/2}$ at 1 are equivalent we replace γ with $\gamma' = [i+1, 1]$. Still $[h_{1/2}, \gamma'] = F_g([f]) \star [g]$. This is done to be sure that the access curve lies outside of the sets in the family K_s as it is required by Corollary 3.5.

Then we continue the process described above for $1/2 \leq s \leq 1$. Finally, K_1 will consists of D^1 and $D_1^2 = \{|\zeta - 2| \leq 1\}$. The mapping h_1 is equal to g on D^1 and to $f(2-\zeta)$. Now it is clear that $[h_1, \gamma'] = [g] \star [f]$.

(5) We start with the compact set K_1 consisting of the interval $I = [0, 1]$ and the unit disk $D_1 = \{|\zeta - 2| \leq 1\}$. The mapping f_1 on K_1 is defined as $g(e^{2\pi it})$ on I and as $g(2-\zeta)$ on D_1. If the access curve $\gamma = [-i, 0]$, then $[f_1, \gamma] = F_g([g])$.

For $0 \leq s \leq 1$ we define compact sets K_s consisting of the intervals $I_s = [0, s]$ and the disks $D_s = \{|\zeta - (1+s)| \leq 1\}$. The mapping f_s is defined as $g(e^{2\pi it})$ on I_s and as $g(e^{2\pi is}(1+s-\zeta))$ on D_s. The pairs (K_s, f_s) form a continuous path and $[f_s, \gamma] = F_g([g])$. Since K_0 consists of the disk $\{|\zeta - 1| \leq 1\}$ and the mapping $f_0(\zeta) = g(1-\zeta)$ we see that $[f_1, \gamma] = [g]$. □

As a consequence we have the following corollary.

Corollary 6.5. *If w_0 and w_1 are two points of W then $\eta_1(W, M, w_0)$ is isomorphic to $\eta_1(W, M, w_1)$.*

7. Finitely connected domains

The main goal of this section is to study $\eta_1(W, \mathbb{C}, w_0)$ when W is a finitely connected domain in $\mathbb{C}$, $M = \mathbb{C}$ and $\Pi(z) = z$. First we consider the case when our domain $W = W' \setminus E$, where W' is a connected and simply connected domain and $E = \{w_1, \cdots, w_m\}$ is a set of distinct points in W'. Let $w_0 \in W$ be a base point.

The fundamental group of W is a free group on m generators. We will fix the set of generators by choosing simple continuous curves $\alpha_j : [0, 1] \to W$, $1 \leq j \leq m$, such that α_j connects w_0 with w_j, never meets E, when $0 \leq t < 1$, and these curves meet each other only at w_0. If we take sufficiently small disjoint disks d_j centered at w_j and a point $\zeta_j \in \partial d_j$, where α_j meets ∂d_j, then the union of these curves and disks d_j with deleted centers is a homotopy retract of W. Therefore, the set of homotopy classes of equivalence of curves $\{\lambda_j, 1 \leq j \leq m\}$, where λ_j is the curve which starts at w_0, follows α_j up to ζ_j, then goes counterclockwise by ∂d_j until ζ_j and then returns to w_0 by α_j, will be the set of generators of $\pi_1(W, w_0)$ which will

be denoted by $\{e_1, \dots, e_m\}$. Clearly, the homotopy classes of λ_j do not depend on the radii of the disks d_j provided that they are sufficiently small.

Let β_j be the curve defined as λ_j from w_0 to ζ_j. For each j we consider the mapping f_j of $K = [0,1] \cup \{|\zeta - 2| \le 1\}$ equal to β_j on $[0,1]$ and a conformal mapping onto d_j on $\{|\zeta-2| \le 1\}$ moving 1 to ζ_j. We will define $[g_j] \in \eta_1(W, \mathbb{C}, w_0)$ as $I_\gamma(f_j)$, where the access curve $\gamma = [-i, 0]$. Then $\iota_1([g_j]) = e_j$.

If $f \in \mathcal{S}_{1,w_0}(\overline{\mathbb{D}}, W, \mathbb{C})$ then $\hat{f}$ takes values $w_1, \dots, w_m$ at finitely many points. The number of these points counted with multiplicity of f at these points will be called the index of f. If $f, g \in \mathcal{S}_{1,w_0}(\overline{\mathbb{D}}, W, \mathbb{C})$ and $[f] = [g]$, then their indexes coincide. We will need the following lemma.

Lemma 7.1. *If $f \in \mathcal{S}_{1,w_0}(\overline{\mathbb{D}}, W, \mathbb{C})$ then $[f] = [f_1] \star \dots \star [f_k]$, where k is equal to the index of f, and the index of each of f_j, $1 \le j \le k$, is 1.*

Proof. Shift f slightly to ensure that any value $w_1, \dots, w_m$ is taken with multiplicity 1. Choose $k-1$ curves α_j dividing $\overline{\mathbb{D}}$ into Jordan domains containing exactly one preimage of the set $\{w_1, \dots, w_m\}$ and apply Proposition 4.3. □

Now we want to describe the h-homotopy classes of mappings with index 1.

Lemma 7.2. *Let $f \in \mathcal{S}_{1,w_0}(\overline{\mathbb{D}}, W, \mathbb{C})$ has index 1 and $\hat{f}$ takes the value w_j. Then there is a loop λ at w_0 such that $[f] = F_\lambda([g_j])$ and $\iota_1([f]) = F_\lambda(e_j)$.*

Proof. By the continuity of the homotopic type we may assume that $\hat{f}$ is defined on $\mathbb{D}_r = \{|\zeta| < r\}$, $r > 1$, maps $\mathbb{D}_r$ into W' and is of index 1 on $\mathbb{D}_r$. We take a small disk d_j centered at w_j so that $\hat{f}^{-1}$ is defined on d_j. Let $d'_j = \hat{f}^{-1}(d_j)$, $\xi_j = \hat{f}^{-1}(w_j)$ and let $\zeta'_j = \hat{f}^{-1}(\zeta_j)$. We connect 1 and ζ'_j by a curve μ in $\mathbb{D} \setminus d'_j$.

Let K be the union of $[0,1]$ and $D = \{|\zeta - 2| \le 1\}$ and the access curve γ to K at 0 is $[-i, 0]$. We define $\phi \in \mathcal{A}(K, \mathbb{D}_r)$ as μ on $[0,1]$ and a conformal mapping of D onto d'_j such that $\phi(1) = \zeta'_j$.

We want to show that $[f] = I_\gamma(f \circ \phi)$. By Mergelyan Theorem we can approximate ϕ by holomorphic mappings on neighborhoods of K as well as we want. So we can find a smooth Jordan domain Ω containing K and meeting γ and a holomorphic mapping $\psi \in \mathcal{A}(\overline{\Omega}, \mathbb{D}_r)$ such that $\psi(\partial\Omega) \subset \mathbb{D}_r \setminus \{\xi_j\}$, ψ takes the value ξ_j only once, $\psi(\zeta_{\Omega,\gamma}) = 1$ and $\{f \circ \psi, \zeta_{\Omega,\gamma}\} = I_\gamma(f \circ \phi)$. Let e be a conformal mapping of $\overline{\mathbb{D}}$ onto $\overline{\Omega}$ such that $e(1) = \zeta_{\Omega,\gamma}$. Then $[\psi \circ e] = \{\psi, \zeta_{\Omega,\gamma}\}$ in $\eta_1(\mathbb{D}_r \setminus \{\xi_j\}, \mathbb{C}, 1)$.

Considering a conformal mapping of $\mathbb{D}_r$ onto itself which maps ξ_j to 0 we see that by Proposition 6.1, Corollary 6.5 and Theorem 5.1 the semigroup $\eta_1(\mathbb{D}_r \setminus \{\xi_j\}, \mathbb{C}, 1) \simeq \mathbb{N}_0$ and the isomorphism is established by mapping $f \in \mathcal{S}_{1,1}(\overline{\mathbb{D}}, \mathbb{D}_r \setminus \{\xi_j\}, \mathbb{C})$ into the index of f. Hence there is a continuous path η_t connecting $\psi \circ e$ and the identity in $\mathcal{S}_{1,1}(\overline{\mathbb{D}}, \mathbb{D}_r \setminus \{\xi_j\}, \mathbb{C})$. Consequently, the path $f \circ \eta_t$ is also continuous and connects $f \circ \psi \circ e$ and f in $\mathcal{S}_{1,w_0}(\overline{\mathbb{D}}, W, \mathbb{C})$. Thus $[f] = I_\gamma(f \circ \phi)$.

Let $h_j \in \mathcal{S}_{1,\zeta_j}(\overline{\mathbb{D}}, W, \mathbb{C})$ be the conformal mapping of $\mathbb{D}$ onto d_j with $h_j(1) = \zeta_j$. Let $\nu = \hat{f} \circ \mu$. Then $[g_j] = F_{\beta_j}([h_j])$ while $[f] = I_\gamma(f \circ \phi) = F_\nu([h_j])$. Hence $[f] = F_\nu(F_{\beta_j^{-1}}([g_j]))$. The concatenation of curves ν and β_j^{-1} is a loop λ at w_0. By Proposition 6.4 $F_\lambda([g_j]) = F_\nu(F_{\beta_j^{-1}}([g_j])) = [f]$.

The equality $\iota_1([f]) = F_\lambda(e_j)$ is evident. □

Combining this lemma with Lemma 7.1 we obtain the following corollary.

Corollary 7.3. *If $f \in \mathcal{S}_{1,w_0}(\overline{\mathbb{D}}, W, \mathbb{C})$ has index k then there are loops μ_j, $1 \leq j \leq k$, at w_0 such that $\iota([f]) = \prod_{j=1}^{k} F_{\mu_j}(e_{n_j})$ and $[f] = \prod_{j=1}^{k} F_{\mu_j}(g_{n_j})$.*

The following lemma is rather crucial.

Lemma 7.4. *Let $f_0, f_1 \in \mathcal{S}_{1,w_0}(\overline{\mathbb{D}}, W, \mathbb{C})$ have index 1 and $\iota_1([f_0]) = \iota_1([f_1])$. Then $[f_0] = [f_1]$.*

Proof. Firstly we note that $\hat{f}_0$ and $\hat{f}_1$ take the same value w_j in W'. Otherwise, $\iota_1([f_0]) \neq \iota_1([f_1])$. By Lemma 7.2 there are loops μ_0 and μ_1 at w_0 such that $[f_0] = F_{\mu_0}([g_j])$ and $[f_1] = F_{\mu_1}([g_j])$. Since $\iota_1([f_0]) = \iota_1([f_1])$ we see that $F_{\mu_0}(e_j) = F_{\mu_1}(e_j)$ or $e_j = F_{\mu_1^{-1}} F_{\mu_0}(e_j)$. If $\mu = \mu_1^{-1}\mu_0$ in the group $\pi_1(W, w_0)$ then $e_j = \mu e_j \mu^{-1}$ or $e_j\mu = \mu e_j$. But the group $\pi_1(W, w_0)$ is free and, therefore, $\mu = e_j^n$.

Since the operator F_ν does not depend on the homotopy class of ν we can write that $F_\mu([g_j]) = F_{[g_j]^n}([g_j])$. By Proposition 6.4(5)

$$[g_j] = F_{[g_j]^{\star n}}([g_j]) = F_{\mu_1^{-1}} F_{\mu_0}([g_j]) = F_{\mu_1^{-1}}([f_0])$$

or $[f_0] = F_{\mu_1}([g_j]) = [f_1]$. □

A semigroup S has the *left cancelation property* (see [CP]) if $ab = ac$ implies $b = c$. Similarly S has the *right cancelation property* if $ac = bc$ implies $a = b$. A semigroup with both left and right cancelation properties is called a *cancelative semigroup*. We will show that $\eta_1(W, \mathbb{C}, w_0)$ is a cancelative semigroup. Cancelation property plays a crucial role in proving the injectivity of ι_1. First we need the following two lemmas.

Lemma 7.5. *Let $f_t \in \mathcal{S}_{1,w_0}(\overline{\mathbb{D}}, W, \mathbb{C})$, $0 \leq t \leq 1$, be a continuous path. Then for each $\varepsilon > 0$ there is a continuous path g_t in $\mathcal{S}_{1,w_0}(\overline{\mathbb{D}}, W, \mathbb{C})$ such that $|g_t(\zeta) - f_t(\zeta)| < \varepsilon$ for $\zeta \in \overline{\mathbb{D}}$, all roots $\zeta_k(t)$ of the equations $g_t(\zeta) = w_j$, $1 \leq j \leq m$, are simple, functions $\zeta_k(t)$ are smooth and $[f_t] = [g_t]$ for all $0 \leq t \leq 1$.*

Proof. There is $\varepsilon > 0$ such that the distance between $f_t(\mathbb{T})$ to ∂W is greater than ε for all t. By considering $f(t, \zeta) = f_t(\zeta)$ as a function from the unit interval to the Banach space $\mathcal{A}(\overline{\mathbb{D}}, \mathbb{C})$ we can use the Weierstrass theorem to approximate it by polynomials of the form $\sum_{j=0}^{k} h_j t^j$, where $h_j \in \mathcal{A}(\overline{\mathbb{D}}, \mathbb{C})$. Then by approximating each h_j by holomorphic polynomials on a neighborhood of $\mathcal{A}(\overline{\mathbb{D}}, \mathbb{C})$, we get a polynomial $h(t, \zeta) = \sum_{k=0}^{N} c_k(t)\zeta^k$ in t and ζ such that $|f(t, \zeta) - h(t, \zeta)| < \varepsilon/2$ on $[0, 1] \times \overline{\mathbb{D}}$. Moreover, we may assume that $h(t, 1) = w_0$ for all t.

For each $c = (c_0, \cdots, c_N) \in \mathbb{C}^{N+1}$ let

$$P_c(\zeta) = \prod_{j=1}^{m} (\sum_{k=0}^{N} c_k \zeta^k - w_j).$$

Let $\zeta_1(c), \cdots, \zeta_{mN}(c)$ be the roots of the polynomial $P_c(\zeta)$. Since all elementary symmetric polynomials of $\zeta_1(c), \cdots, \zeta_{mN}(c)$ are holomorphic, the discriminant $\Delta(c) = \prod_{i<j} (z_i(c) - z_j(c))^2$ of $P_c(\zeta)$ is holomorphic on $\mathbb{C}^{N+1}$. The set $D = \{\Delta = 0\}$ is analytic of dimension N in $\mathbb{C}^{N+1}$ and for all $c \in \mathbb{C}^{N+1} \setminus D$ the roots of $P_c(\zeta)$ are distinct.

Consider the hypersurface $L = \{(c_0, \cdots, c_N) | \sum_{k=0}^{N} c_k = w_0\}$, where w_0 is the base point in W. Since $\Delta(c) \neq 0$ for $c = (0, w_0, 0, \cdots, 0) \in L$, $D \cap L$ is an analytic subset of dimension $N - 1$ in L.

The curve $c(t) = (c_0(t), \dots, c_N(t))$ is analytic and either intersects D at finite number of points or completely lies in the $D \cap L$. In the second case we can find a vector a with $\|a\| < \varepsilon/2$, $\sum_{j=0}^{N} a_j = 0$ and $\Delta(c(0) + a) \neq 0$. So, by replacing $c(t)$ by $c(t) + a$ we can assume that $c(t)$ intersects D at a finite number of points. Since the Hausdorff measure $\mathcal{H}_{2N-1}(D \cap L) = 0$, for each intersection point z there is a neighborhood U of z such that $U \setminus D$ is connected. So, in a sufficiently small neighborhood of each intersection point we can smoothly modify the curve $c(t)$ so that it will lie in $L \setminus D$ and if $g(t,\zeta) = P_{c(t)}(\zeta)$ then $|f(t,\zeta) - g(t,\zeta)| < \varepsilon$ on $[0,1] \times \overline{\mathbb{D}}$. Now $g(t,\zeta) = g_t(\zeta)$ gives us a homotopy with simple roots $\zeta_k(t)$ of the equations $g_t(\zeta) = w_j$, $1 \le j \le m$. Since the roots are simple by the implicit function theorem the functions $\zeta_k(t)$ are smooth.

Since the homotopy $sf_t(\zeta) + (1-s)g_t(\zeta)$, $0 \le s \le 1$, connects f_t and g_t in $\mathcal{S}_{1,w_0}(\overline{\mathbb{D}}, W, \mathbb{C})$ we see that $[f_t] = [g_t]$ for all $0 \le t \le 1$. $\square$

The proof of the lemma below follows the same line of argument as that in the proof of Assertion 2 in the proof of [Sl, Lemma 2.1].

Lemma 7.6. *Let $\zeta_k(t)$, $1 \le k \le n$, are smooth mappings of $[0,1]$ into $\mathbb{D}$ such that $\zeta_i(t) \neq \zeta_j(t)$ when $i \neq j$ and $0 \le t \le 1$. Then there is a C^∞ mapping $\Phi : \overline{\mathbb{D}} \times [0,1] \to \overline{\mathbb{D}}$ such that $\Phi_t(\zeta) = \Phi(\zeta,t)$ is a diffeomorphism from $\overline{\mathbb{D}}$ onto itself for each t, $\Phi_t(\zeta) = \zeta$ when $|\zeta| = 1$ and $\Phi_t(\zeta_j(0)) = \zeta_j(t)$ for $j = 1, \dots, n$.*

Proof. By Whitney extension theorem (see [N, Theorem 1.5.6]) we can find a complex valued C^∞-function $F(t,\zeta)$ on $[0,1] \times \mathbb{C}$ such that $F(t,\zeta_j(t)) = \partial\zeta_j(t)/\partial t$ for $0 \le t \le 1$, $j = 1, \cdots, n$. Replacing F with the product $F\phi$, where ϕ is a C^∞-function with $\phi = 1$ on $\cup_{j=1}^{n} \{(t,\zeta_j(t)) : 0 \le t \le 1\}$ and $\phi = 0$ for $|\zeta| \ge 1$, we can make $F(t,\zeta) = 0$ for $|\zeta| \ge 1$. Then by standard existence and uniqueness theorems for ordinary differential equations, the initial value problem $\partial x/\partial t(t) = F(t, x(t))$, $x(0) = \zeta$, $0 \le t \le 1$, has a unique solution $x(t,\zeta)$. Since $F(t,\zeta)$ is smooth, this solution is smooth on $[0,1] \times \mathbb{C}$.

Now define a mapping $\Phi : \mathbb{C} \times [0,1] \to \mathbb{C}$ by $\Phi(\zeta,t) = x(t,\zeta)$. Then for each $0 \le t \le 1$, Φ_t is a diffemorphism and since the related initial value problem has a unique solution we have $\Phi(\zeta_j(0),t) = \zeta_j(t)$ for $j = 1, \cdots, n$. Also note that $\Phi(\zeta,t) = \zeta$ for all $0 \le t \le 1$ when $|\zeta| \ge 1$. So, the restriction of Φ to $\overline{\mathbb{D}} \times [0,1]$ has desired properties. $\square$

Now we have all the necessary tools to prove the left and right cancelation properties.

Proposition 7.7. *Let $f, g_0, g_1 \in \mathcal{S}_{1,w_0}(\overline{\mathbb{D}}, W, \mathbb{C})$. Then $[f] \star [g_0] = [f] \star [g_1]$ implies $[g_0] = [g_1]$ and $[g_0] \star [f] = [g_1] \star [f]$ implies $[g_0] = [g_1]$.*

Proof. By Corollary 7.3 it suffices to prove the left cancelation property only when the index of f is 1 and we assume that $\hat{f}$ takes value w_1. Consider the compact set $K \subset \mathbb{C}$ from the definition the $\star$ operation which is the union of $K_1 = \{|\zeta - 1| \le 1\}$ and $K_2 = \{|\zeta + 1| \le 1\}$ and let the access curve be $\gamma(t) = -it$, $0 \le t \le 1$. Pick up a smooth simply connected domain D symmetric with respect to the axes with $K \subset D$ and intersecting γ at $\zeta_0 = -it_0$ and and Φ, ε-approximations $(\overline{D}, \widetilde{h}_0)$ and $(\overline{D}, \widetilde{h}_1)$ with respect to γ of (K, h_{f,g_0}) and (K, h_{f,g_1}) respectively such that $\{\widetilde{h}_0, \zeta_0\} = [f] \star [g_0]$ and $\{\widetilde{h}_1, \zeta_0\} = [f] \star [g_1]$. If $D_1 = D \cap \{\mathbf{Re}\,\zeta > 0\}$ and $D_2 = D \cap \{\mathbf{Re}\,\zeta < 0\}$ we may assume that $\{\widetilde{h}_0|_{D_1}, \zeta_0\} = [f]$, $\{\widetilde{h}_0|_{D_2}, \zeta_0\} = [g_0]$,

$\{\widetilde{h}_1|_{D_1}, \zeta_0\} = [f]$ and $\{\widetilde{h}_1|_{D_2}, \zeta_0\} = [g_1]$. We also may assume that the roots of all equation $\widetilde{h}_0 = w_j$ and $\widetilde{h}_1 = w_j$ are simple, none of them lies on $[\zeta_0, -\zeta_0]$, the index of $\widetilde{h}_0$ and $\widetilde{h}_1$ on D_1 is 1 and $\widetilde{h}_0(\zeta_1) = \widetilde{h}_1(\zeta_1) = w_1$ at some $\zeta_1 \in D_1$.

Let Ψ be a conformal mapping of $\mathbb{D}$ onto D such that $\Psi(0) = 0$, $\Psi(1) = \zeta_0$ and $\Psi(-1) = -\zeta_0$. By the symmetry Ψ maps $[1, -1]$ onto $[\zeta_0, -\zeta_0]$, $\mathbb{D}^+ = \mathbb{D} \cap \{\mathbf{Im}\,\zeta > 0\}$ onto D_1 and $\mathbb{D}^- = \mathbb{D} \cap \{\mathbf{Im}\,\zeta < 0\}$ onto D_2. We set $h_0 = \widetilde{h}_0 \circ \Psi$ and $h_1 = \widetilde{h}_1 \circ \Psi$. By the definition of the homotopic type there is an h-homotopy $h_t \in \mathcal{S}_{1,w_0}(\overline{\mathbb{D}}, W, \mathbb{C})$ connecting h_0 and h_1. By Lemma 7.5 we may assume that that the roots $\zeta_k(t)$, $1 \le k \le n$, of all equation $h_t = w_j$, $1 \le j \le m$, are simple for all $t \in [0, 1]$ and the curves ζ_k are smooth. Hence there is a mapping $\Phi : \overline{\mathbb{D}} \times [0, 1] \to \overline{\mathbb{D}}$ satisfying the conclusion of Lemma 7.6.

The curve $\alpha_t = \Phi_t([-1, 1])$, connecting 1 and -1 in $\overline{\mathbb{D}}$, divides $\mathbb{D}$ into Jordan domains $G_t^1 = \Phi_t(\mathbb{D}^+)$ and $G_t^2 = \Phi_t(\mathbb{D}^-)$. Note that if $\zeta \in \alpha_t$ then $h_t(\zeta) \neq w_j$ for any $1 \le j \le m$. Since Φ_t is a smooth family of diffeomorphisms the families G_t^1 and G_t^2 are Radó continuous. Hence $(G_t^1, h_t|_{G_t^1})$ and $(G_t^2, h_t|_{G_t^2})$ are continuous paths in $\mathcal{S}^*(\mathbb{C}, W, \mathbb{C})$ and, therefore, $\{h_t|_{G_t^1}, 1\} = [f]$ and $\{h_t|_{G_t^2}, 1\} = [g_0]$ for all $t \in [0, 1]$.

We can slightly shift the curve α_1 so that the homotopic types will not change and the intersection of G_1^1 with $\mathbb{D}^-$ consists of finitely many domains Ω_j^- bounded by parts of α_1 and $[-1, 1]$. Analogously the intersection of G_1^2 with $\mathbb{D}^+$ also consists of finitely many domains Ω_j^+ bounded by parts of α_1 and $[-1, 1]$. Each of them is a Jordan domain being bounded by simple curves. The restriction of h_1 to $\mathbb{D}^+$ is h-homotopic to f and, therefore, contains exactly one preimage of the set E under the mapping h_1, namely, of w_1. By construction the domain G_1^1 also contains exactly one preimage of the set E under the mapping h_1 and it can be only w_1. Hence only at most one of the domains Ω_j^+ may contain a preimage of w_j under the mapping h_1 and if this happens then exactly one of Ω_j^- contains a preimage of w_j and vice versa. Moreover, this is a preimage of w_1.

If domains Ω_j^- do not contain a preimage of w_1 under the mapping h_1, then the restrictions of h_1 to domains Ω_j^- and Ω_j^+ are homotopic to a constant mapping. Let $G = G_1^2 \cap \mathbb{D}^-$ and let q be the restriction of h_1 to $\overline{G}$. By Proposition 4.3 we can erase in G_1^2 domains Ω_j^+ to get $\{h_1|_{G_1^2}, 1\} = \{q, 1\} = [g_0]$. Also in $\overline{\mathbb{D}}^-$ we can erase Ω_j^- to get $[g_1] = \{h_1|_{\mathbb{D}^-}, 1\} = \{q, 1\}$. Hence $[g_0] = [g_1]$.

If, say, Ω_1^- contains a preimage of w_1 under the mapping h_1, then only one domain Ω_j^+, say, Ω_1^+ also contains exactly one preimage of w_1 under the mapping h_1. Let us denote by L^+ the compact set consisting of the path starting at 1 and following the boundary of G_1^1 until it reaches Ω_1^+ and the set $\overline{\Omega}_1^+$. Let p^+ will be restriction of h_1 to L. By Proposition 4.3 $[g_0] = \{h_1|_{G_1^2}, 1\} = I_\gamma(p^+) \star \{q, 1\}$, where $\gamma = [1, 2]$.

Let L^- be the compact set consisting of the path starting at 1 and following $[1, -1]$ until it reaches Ω_1^- and the set $\overline{\Omega}_1^-$. Let p^- will be restriction of h_1 to L^-. By Proposition 4.3 $[g_1] = \{h_1|_{\mathbb{D}^-}, 1\} = I_\gamma(p^-) \star \{q, 1\}$. Since $\iota_1([g_0]) = \iota_1([g_1])$ we see that $\iota_1(I_\gamma(p^+)) = \iota_1(I_\gamma(p^-))$. By Lemma 7.4 $I_\gamma(p^+) = I_\gamma(p^-)$ and, consequently, $[g_0] = [g_1]$.

Similarly we can prove the right cancelation property. $\square$

We need the notion of a *nested* word in $\eta_1(W, \mathbb{C}, w_0)$. It is defined by induction. A nested word of level 0 is the word of the form $g_{j_1}^{k_1} \star \cdots \star g_{j_n}^{k_n}$, where $g_{j_{l+1}} \neq g_{j_l}$ and all $k_j > 0$. A nested word of level n is a word

$$F_{\lambda_1}(B_1) \star \cdots \star F_{\lambda_n}(B_m), \tag{1}$$

where the words $B_1, \ldots, B_m$ are nested words of level at most $n-1$. By Corollary 7.3 every word in $\eta_1(W, \mathbb{C}, w_0)$ can be written as a nested word of level 1.

Given a nested word $g_{j_1}^{k_1} \star \cdots \star g_{j_n}^{k_n}$ of level 0 its *precise copy* in $\pi_1(W, w_0)$ is the word $e_{j_1}^{k_1} \cdots e_{j_n}^{k_n}$. Clearly, this word is reduced, i.e., no cancelations are possible. If a nested word is of level n and has the form as in (1) then its precise copy is

$$\lambda_1 \widetilde{B}_1 \lambda_1^{-1} \cdots \lambda_n \widetilde{B}_n \lambda_n^{-1},$$

where the words $\widetilde{B}_1, \ldots, \widetilde{B}_m$ are precise copies of words $B_1, \ldots, B_m$ respectively.

For nested words we can prove the following lemma.

Lemma 7.8. *Every word in $\eta_1(W, \mathbb{C}, w_0)$ can be written as a nested word whose precise copy is reduced.*

Proof. To prove the lemma we will show that if a precise copy of a nested word admits cancelations then we can rewrite it as a nested word such that the length of the precise copy is decreased at least by 2. We will do it considering four possible cases.

1) Suppose that a cancelation is possible at level 1, i.e., somewhere in the word we have $\lambda g_{j_1}^{k_1} \star \cdots \star g_{j_n}^{k_n} \lambda^{-1}$ and $\lambda = \mu g_{j_1}^{-1}$. Then using in turn properties (1), (4) and (5) from Proposition 6.4 the latter word can be rewritten as

$$\begin{aligned} &F_\mu \left(F_{g_{j_1}^{-1}}(g_{j_1}^{k_1}) \star F_{g_{j_1}^{-1}}(g_{j_2}^{k_2} \star \cdots \star g_{j_n}^{k_n}) \right) \\ &= F_\mu \left(g_{j_1}^{k_1} \star F_{g_{j_1}^{-1}}(g_{j_2}^{k_2} \star \cdots \star g_{j_n}^{k_n}) \right) = F_\mu \left(g_{j_1}^{k_1 - 1} \star g_{j_2}^{k_2} \star \cdots \star g_{j_n}^{k_n} \star g_{j_1} \right) \end{aligned}$$

and we see that the length of the precise copy decreases by 2.

2) Suppose the cancelation is possible in operators F_λ, i.e. λ has adjacent g_j and g_j^{-1} as factors. Then they can be canceled by Proposition 6.4(2) and again the length of the precise copy decreases by 2.

3) Suppose the cancelation is possible between adjacent factors, i.e., somewhere in the word we have $F_\lambda(B_1) \star F_\mu(B_2)$ and $\lambda = g_j \lambda_1$ while $\mu = g_j \mu_1$. Then by Proposition 6.4 (1)

$$F_\lambda(B_1) \star F_\mu(B_2) = F_{g_j}(F_{\lambda_1}(B_1)) \star F_{g_j}(F_{\mu_1}(B_2)) = F_{g_j}\left(F_{\lambda_1}(B_1) \star F_{\mu_1}(B_2)\right)$$

and we see that the length of the precise copy decreases by 2. The case when $\lambda = g_j^{-1} \lambda_1$ while $\mu = g_j^{-1} \mu_1$ can be considered analogously.

4) Suppose the cancelation occurs at adjacent levels, i.e., somewhere in the word we have $F_\lambda(F_{\mu_1}(B_1) \star \cdots \star F_{\mu_n}(B_n))$ and $\lambda = \lambda_1 g_j$ while $\mu_1 = g_j^{-1} \mu_{12}$. Then by Proposition 6.4 (1)

$$\begin{aligned} &F_\lambda(F_{\mu_1}(B_1) \star \cdots \star F_{\mu_n}(B_n)) \\ &= F_{\lambda_1} F_{g_j} \left(F_{g_j^{-1}} F_{\mu_{12}}(B_1) \star F_{\mu_2}(B_2) \star \cdots \star F_{\mu_n}(B_n) \right) \\ =&F_{\lambda_1} F_{g_j} \left(F_{g_j^{-1}} F_{\mu_{12}}(B_1) \right) \star F_{\lambda_1} F_{g_j} \left(F_{\mu_2}(B_2) \star \cdots \star F_{\mu_n}(B_n) \right) \\ =&F_{\lambda_1} \left(F_{\mu_{12}}(B_1) \star F_{g_j} \left(F_{\mu_2}(B_2) \star \cdots \star F_{\mu_n}(B_n) \right) \right) \end{aligned}$$

and again the length of the precise copy decreases by 2. The cases when $\lambda = \lambda_1 g_j^{-1}$ while $\mu_1 = g_j\mu_{12}$ or cancelation occurs between λ^{-1} and μ_n can be considered analogously. $\square$

Now we can prove the following proposition.

Proposition 7.9. *If $f_0, f_1 \in \mathcal{S}_{w_0}(\overline{\mathbb{D}}, W, \mathbb{C})$ and $\iota_1([f_0]) = \iota_1([f_1])$, then $[f_0] = [f_1]$.*

Proof. We denote by $l([f])$ the length of the reduced word $\iota_1([f])$ and call it the length of f. The proof will be the induction by $l([f])$. If $l([f_0]) = 1$ then $l([f_1]) = 1$ and this means that f_0 and f_1 have index 1. By Lemma 7.4 $[f_0] = [f_1]$.

Now suppose that we proved the theorem for the length less or equal to $k-1$ and let $l([f_0]) = l([f_1]) = k$. Suppose also that the reduced word for $\iota_1([f_0])$ contains e_j^{-1} for some $1 \le j \le m$, i.e. $\iota_1([f_0]) = \lambda e_j^{-1}\mu$. Consider $[h_0] = F_{\lambda^{-1}}([f_0])$ and $[h_1] = F_{\lambda^{-1}}([f_1])$. Then $\iota_1([h_0]) = \iota_1([h_1]) = e_j^{-1}\mu\lambda$. Hence $l([h_0]) \le k$ and $l([g_j] \star [h_0]) \le k-1$. By the induction assumption $[g_j] \star [h_0] = [g_j] \star [h_1]$ and by the left cancelation property from Proposition 7.7 $[h_0] = [h_1]$. Thus $[f_0] = [f_1]$.

If $\iota_1([f_0])$ does not contain g_j^{-1} for any $1 \le j \le m$, i.e., $\iota_1([f_0]) = e_{j_1}^{k_1} \cdots e_{j_n}^{k_n}$, where $e_{j_{l+1}} \ne e_{j_l}$ and all $k_j > 0$. Then by Lemma 7.8 we can rewrite $[f_0]$ as a nested word whose precise copy is reduced and is equal to $\iota_1([f_0])$. Then all operators F_λ in this word are identities and we see that $[f_0] = g_{j_1}^{k_1} \star \cdots \star g_{j_n}^{k_n}$. The same is true for $[f_1]$ and the proposition is proved. $\square$

Now we can describe the semigroup $\eta_1(W, \mathbb{C}, w_0)$ when W is a general finitely connected domain. Let $\widetilde{W} = W' \setminus H_1 \cup \cdots \cup H_m$, where W' is a connected and simply connected domain and $H_1, \dots, H_m$ are disjoint connected compact sets in W'. We will fix the set of generators in $\pi_1(\widetilde{W}, w_0)$ by choosing points $w_j \in \partial H_j$ and simple continuous curves $\widetilde{\alpha}_j : [0,1] \to W$, $1 \le j \le m$, such that $\widetilde{\alpha}_j$ connects w_0 with w_j, never meets $H = H_1 \cup \cdots \cup H_m$ when $0 \le t < 1$ and these curves meet each other only at w_0. We take smooth disjoint Jordan curves $C_j \subset \widetilde{W}$, whose interiors contain only H_j, and the points ζ_j, where $\widetilde{\alpha}_j$ meets C_j last time. Let $\widetilde{\lambda}_j$ be a curve which starts at w_0, follows $\widetilde{\alpha}_j$ up to ζ_j, then goes counterclockwise by C_j until ζ_j and then returns to w_0 by $\widetilde{\alpha}_j$. Then the set of homotopy equivalence classes of curves $\{\widetilde{\lambda}_j, 1 \le j \le m\}$ will be the set of generators in $\pi_1(\widetilde{W}, w_0)$ which will be denoted by $\{\widetilde{e}_1, \dots, \widetilde{e}_m\}$. Clearly, the homotopy classes of $\widetilde{\lambda}_j$ do not depend on the choice of C_j provided they are chosen sufficiently close to H_j.

For each j we consider the mapping $\widetilde{f}_j$ of $K = [0,1] \cup \{|\zeta - 2| \le 1\}$ defined as $\widetilde{\alpha}_j$ from w_0 to ζ_j on $[0,1]$ and a conformal mapping of $\{|\zeta - 2| \le 1\}$ onto the bounded domain C_j^i which has C_j as its boundary such that $\widetilde{f}_j(1) = \zeta_j$. We will define $[\widetilde{g}_j] \in \eta_1(W, \mathbb{C}, w_0)$ as $I_\gamma(\widetilde{f}_j)$, where the access curve $\gamma = [-i, 0]$. Then $\iota_1([\widetilde{g}_j]) = \widetilde{e}_j$.

Let ψ be a homeomorphism of $\widetilde{W}$ onto $W = W' \setminus \{w_1, \dots, w_m\}$. We assume that ψ is a continuous mapping of W' onto itself collapsing each H_j to w_j. Let $\alpha_j = \psi \circ \widetilde{\alpha}_j$ and let $\{e_j\}$ and $\{[g_j]\}$, $1 \le j \le m$, be the generators of $\pi_1(W, w_0)$ and $\eta_1(W, \mathbb{C}, w_0)$ respectively defined at the beginning of this section for W using the curves α_j. Then it is easy to see that the isomorphism ψ_* between $\pi_1(\widetilde{W}, w_0)$ and $\pi_1(W, w_0)$ generated by ψ sends $\widetilde{e}_j$ to e_j.

Theorem 7.10. *If $\widetilde{W}$ is a finitely connected domain in $\mathbb{C}$, $w_0 \in \widetilde{W}$, then the mapping $\iota_1 : \eta_1(\widetilde{W}, \mathbb{C}, w_0) \to \pi_1(\widetilde{W}, w_0)$ is an imbedding and the semigroup $\eta_1(\widetilde{W}, \mathbb{C}, w_0)$ is isomorphic to the minimal subsemigroup of $\pi_1(\widetilde{W}, w_0)$ containing $\{\widetilde{e}_j, 1 \le j \le m\}$ and invariant with respect to the inner automorphisms.*

Proof. If $f_0, f_1 \in \mathcal{S}_{1,w_0}(\overline{\mathbb{D}}, \widetilde{W}, \mathbb{C})$ then $f_0, f_1 \in \mathcal{S}_{1,w_0}(\overline{\mathbb{D}}, W, \mathbb{C})$ and if $[f_0] = [f_1]$ in $\eta_1(\widetilde{W}, \mathbb{C}, w_0)$ then $[f_0] = [f_1]$ in $\eta_1(W, \mathbb{C}, w_0)$. Hence we have a mapping $\Lambda : \eta_1(\widetilde{W}, \mathbb{C}, w_0) \to \eta_1(W, \mathbb{C}, w_0)$.

Let us show that Λ is injective. Suppose that $[f_0] = [f_1]$ in $\eta_1(W, \mathbb{C}, w_0)$, i.e., there is an h-homotopy $f_t \in \mathcal{S}_{1,w_0}(\overline{\mathbb{D}}, W, \mathbb{C})$, $0 \le t \le 1$, connecting f_0 and f_1. There are some closed disjoint disks d_j centered at w_j, $1 \le j \le m$, such that $f_t \in \mathcal{S}_{1,w_0}(\overline{\mathbb{D}}, \hat{W}, \mathbb{C})$, $0 \le t \le 1$, where $\hat{W} = W' \setminus d_1 \cup \dots \cup d_m$. We choose them so small that $d_j \subset C^i_j$ and there is a conformal mapping q_j of $C^i_j \setminus d_j$ onto an annulus $A = \{r < |\zeta| < 1\}$.

We may assume that the curves C_j has been chosen so that the mappings $f_0, f_1 \in \mathcal{S}_{1,w_0}(\overline{\mathbb{D}}, U, \mathbb{C})$, where $U = W' \setminus C$ and C is the closure of $C^i_1 \cup \dots \cup C^i_m$. We denote by B_{js} the preimages of the circles $\{|\zeta| = s\}$ under the mapping q_j. Let $U_s = W' \setminus B^i_{1s} \cup \dots \cup B^i_{ms}$. There are numbers $r < s_0 < s_1 < 1$ such that for all $t \in [0,1]$ the mappings f_t map $\mathbb{T}$ into U_{s_0} and $U_{s_1} \subset \widetilde{W}$.

Let $s(t)$ be the maximal number of those s for which f_t maps $\mathbb{T}$ into U_s and $f_t(\mathbb{T})$ meets $W' \setminus U_{s_1}$. If $f_t(\mathbb{T})$ does not meet $W' \setminus U_{s_1}$ we set $s(t) = s_1$. We define a homeomorphism Q_t of W' onto itself in the following way. If $z \in U$ or $z \in d_j$ or $s(t) = s_1$ then $Q_t(z) = z$. If $s(t) < s_1$ and $z \in C^i_j \setminus d_j$ then $Q_t(z) = q_j^{-1}(p_t(q_j(z)))$, where $p_t(xe^{i\alpha}) = a_t(x)e^{i\alpha}$ and a_t is an increasing function made from two linear functions so that $a_t(r) = r$, $a_t(s(t)) = s_1$ and $a(1) = 1$. Since $s(t) > s_0 > r$ this definition is correct.

A simple geometric argument shows that Q_t is quasiconformal (see [AIM]) and there is $k < 1$ such that its Beltrami coefficient

$$\mu_t(z) = \frac{\partial Q_t}{\partial \bar{z}}(z) / \frac{\partial Q_t}{\partial z}(z) \tag{2}$$

is less than k by the absolute value for all $t \in [0,1]$. Moreover, $Q_t(z)$ is continuous in t and if $t \to t_0$ then $\mu_t \to \mu_{t_0}$ almost everywhere in W'.

Let $h_t = Q_t \circ f_t$. Then

$$\frac{\partial h_t}{\partial \zeta}(\zeta) = \frac{\partial Q_t}{\partial z}(f_t(\zeta)) \frac{\partial f_t}{\partial \zeta}(\zeta) \text{ and } \frac{\partial h_t}{\partial \bar{\zeta}}(\zeta) = \frac{\partial Q_t}{\partial \bar{z}}(f_t(\zeta)) \overline{\frac{\partial f_t}{\partial \zeta}(\zeta)}.$$

By (2) we get

$$\frac{\partial h_t}{\partial \bar{\zeta}}(\zeta) = \mu_t(f_t(\zeta)) \frac{\partial Q_t}{\partial z}(f_t(\zeta)) \overline{\frac{\partial f_t}{\partial \zeta}(\zeta)} = \mu_t(f_t(\zeta)) \frac{\overline{(f_t)_\zeta}}{(f_t)_\zeta} \frac{\partial h_t}{\partial \zeta}(\zeta).$$

So, h_t is a continuous family of quasiregular mappings of $\overline{\mathbb{D}}$ into W' with Beltrami coefficients $\nu_t(\zeta) = \mu_t(f_t(\zeta)) \frac{\overline{(f_t)_\zeta}}{(f_t)_\zeta}$. Moreover, $h_t(\mathbb{T}) \subset U_{s_1}$ and if $t \to t_0$ then $\nu_t \to \mu_{t_0}$ almost everywhere in W'.

By [AIM, Theorem 9.0.3] there is a homeomorphism ψ_t of $\overline{\mathbb{D}}$ onto itself satisfying the equation

$$\frac{\partial \psi_t}{\partial \bar{\eta}} = \nu_t(\eta) \frac{\partial \psi_t}{\partial \eta}$$

and such that $\psi_t(0) = 0$ and $\psi_t(1) = 1$. If $\phi_t = \psi_t^{-1}$ then by formula (2.51) in [AIM]

$$\frac{\partial \phi_t}{\partial \bar{\xi}} = -\nu_t(\phi_t)\overline{\frac{\partial \phi_t}{\partial \bar{\xi}}}.$$

Hence

$$\frac{\partial h_t \circ \phi_t}{\partial \bar{\xi}} = \frac{\partial h_t}{\partial \zeta}(\phi_t(\xi))\frac{\partial \phi_t}{\partial \bar{\xi}} + \frac{\partial h_t}{\partial \bar{\zeta}}(\phi_t(\xi))\overline{\frac{\partial \phi_t}{\partial \xi}} = 0$$

and this means that the mappings $h_t \circ \phi_t$ are holomorphic and by the lemma below the path $h_t \circ \phi_t$ is continuous in $\mathcal{S}_{1,w_0}(\overline{\mathbb{D}}, \widetilde{W}, \mathbb{C})$.

Note that Q_0 and Q_1 are identities. Hence $\mu_0 \equiv \mu_1 \equiv \nu_0 \equiv \nu_1 \equiv 0$. Hence ψ_0, ψ_1, ϕ_0 and ϕ_1 are identities and since by the definition $h_0 \circ \phi_0 = Q_0 \circ f_0 \circ \phi_0$ we see that $h_0 = h_0 \circ \phi_0 = f_0$. By the same reason $h_1 = h_1 \circ \phi_1 = f_1$. Hence $[f_0] = [f_1]$ in $\eta_1(\widetilde{W}, \mathbb{C}, w_0)$ and we see that Λ is injective. Clearly, $\Lambda([\widetilde{g}_j]) = [g_j]$, where $[g_j]$ were defined at the beginning of this section and, therefore, Λ is an isomorphism.

If $f_0, f_1 \in \mathcal{S}_{1,w_0}(\overline{\mathbb{D}}, \widetilde{W}, \mathbb{C})$ and $\iota_1([f_0]) = \iota_1([f_1])$ in $\pi_1(\widetilde{W}, w_0)$, then $\iota_1([f_0]) = \iota_1([f_1])$ in $\pi_1(W, w_0)$. By Proposition 7.9 $[f_0] = [f_1]$ in $\eta_1(W, \mathbb{C}, w_0)$ and by the previous result $[f_0] = [f_1]$ in $\eta_1(\widetilde{W}, \mathbb{C}, w_0)$. Hence the mapping $\iota_1 : \eta_1(\widetilde{W}, \mathbb{C}, w_0) \to \pi_1(\widetilde{W}, w_0)$ is an isomorphism of $\eta_1(\widetilde{W}, \mathbb{C}, w_0)$ onto a subsemigroup $G \subset \pi_1(\widetilde{W}, w_0)$.

If μ is a loop in $\widetilde{W}$ starting at w_0 and $[f] \in \eta_1(\widetilde{W}, \mathbb{C}, w_0)$, then $F_\mu([f]) \in \eta_1(\widetilde{W}, \mathbb{C}, w_0)$ and since $\iota_1(F_\mu([f])) = \mu\iota_1([f])\mu^{-1}$ we see that G is invariant with respect to the inner automorphisms. On the other hand, given $f \in \mathcal{S}_{1,w_0}(\overline{\mathbb{D}}, \widetilde{W}, \mathbb{C})$ by Corollary 7.3 $\Lambda([f]) = \prod_{j=1}^k F_{\lambda_j}([g_j])$. Since we can find loops $\mu_j \in \widetilde{W}$ homotopic to λ_j in W we can write $\Lambda([f]) = \prod_{j=1}^k F_{\mu_j}([g_j])$. But then $[f] = \prod_{j=1}^k F_{\mu_j}([\widetilde{g}_j])$ and we see that G is the minimal subsemigroup of $\pi_1(\widetilde{W}, w_0)$ containing $\{\widetilde{e}_j, 1 \le j \le m\}$ and invariant with respect to the inner automorphisms.. $\square$

Lemma 7.11. *Suppose we have a sequence of Beltrami coefficients $\{\mu_n\}$ such that $|\mu_n| \le k < 1$ for all n and almost every $z \in \mathbb{D}$ and the pointwise limit $\mu(z) = \lim_{n\to\infty} \mu_n(z)$ exists almost everywhere in $\mathbb{D}$. Let ϕ^n be the normalized ($\phi^n(0) = 0$ and $\phi^n(1) = 1$) solution to $\phi^n_{\bar{z}} = \mu^n(z)\phi^n_z$ which is a homeomorphism of $\overline{\mathbb{D}}$. Then the limit $\phi(z) = \lim_{n\to\infty} \phi^n(z)$ exists, the convergence is uniform on $\overline{\mathbb{D}}$ and ϕ solves the Beltrami equation $\phi_{\bar{z}} = \mu(z)\phi_z$.*

Proof. Each $\phi^n : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ extends to a K-quasiconformal homeomorphis of $\mathbb{C}$ onto $\mathbb{C}$ defined by $\Phi^n(z) = \phi^n(z)$ when $|z| \le 1$ and $\Phi^n(z) = 1/\overline{\phi^n(1/\bar{z})}$ when $|z| > 1$. Note that $\Phi^n_{\bar{z}} = \widetilde{\mu}_n(z)\Phi^n_z$, where $\widetilde{\mu}_n(z) = \mu_n(z)$ when $|z| < 1$ and

$$\widetilde{\mu}_n(z) = \frac{z^2\overline{\mu_n(1/\bar{z})}}{\bar{z}^2}$$

when $|z| > 1$. Thus $\mu_n \to \mu$ almost everywhere in $\mathbb{C}$.

By [AIM, Lemma 5.3.5] the limit $\Phi(z) = \lim_{n\to\infty} \Phi^n(z)$ exists, the convergence is uniform on compact sets in $\mathbb{C}$ and Φ solve the equation $\Phi_{\bar{z}} = \widetilde{\mu}(z)\Phi_z$. By [AIM, Theorem 3.9.4] Φ is a non-constant K-quasiconformal homeomorphism of $\mathbb{C}$ onto itself. Since $\Phi(\overline{\mathbb{D}}) = \overline{\mathbb{D}}$ the lemma is proved. $\square$

Finally, we will prove the Oka principle for $\mathcal{S}_{1,w_0}(\overline{\mathbb{D}}, W, \mathbb{C})$ when W is a domain in $\mathbb{C}$.

Theorem 7.12. *Let $W \subset \mathbb{C}$ be a domain. Then the mapping ι_1 is an imbedding and the semigroup $\eta_1(W, \mathbb{C}, w_0)$ is cancelative.*

Proof. Suppose that $f_0, f_1 \in \mathcal{S}_{1,w_0}(\overline{\mathbb{D}}, W, \mathbb{C})$ and $\iota_1([f_0]) = \iota_1([f_1])$. hence there is a continuous mapping $F : [0,1] \times \mathbb{T} \to W$ such that $F(t,1) = w_0$, $F(0,\zeta) = f_0(\zeta)$ and $F(1,\zeta) = f_1(\zeta)$. There is $\delta > 0$ such that the distance in the spherical metric from $F(t,\zeta)$ to $L = \mathbb{CP}^1 \setminus W$ is always greater than δ. If K is the closed $\delta/2$-neighborhood of L the $\mathbb{C}P^1 \setminus K$ is the union of bounded finitely connected subdomains in W. Let us denote by W' one of them which contains $F([0,1] \times \mathbb{T})$. By Theorem 7.10 $[f_0] = [f_1]$ in $\mathcal{S}_{1,w_0}(\overline{\mathbb{D}}, W', \mathbb{C})$. Hence $[f_0] = [f_1]$ in $\mathcal{S}_{1,w_0}(\overline{\mathbb{D}}, W, \mathbb{C})$ and we see that ι_1 is an imbedding.

If $[f] \star [g_0] = [f] \star [g_1]$ then

$$\iota_1([f] \star [g_0]) = \iota_1([f] \star [g_1]) = \iota_1([f]) \cdot \iota_1([g_0]) = \iota_1([f]) \cdot \iota_1([g_1])$$

and we see that $\iota_1([g_0]) = \iota_1([g_1])$. By the argument above $[g_0] = [g_1]$. □

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